CALT-TH-2015-030

New Formulation of the Type IIB Superstring Action in $\mathbf{AdS_5 \times S^5}$

John H. Schwarz[1]
Walter Burke Institute for Theoretical Physics
California Institute of Technology 452-48
Pasadena, CA 91125, USA

Abstract

Previous studies of the type IIB superstring in an $AdS_5 \times S^5$ background are based on a description of the superspace geometry as the quotient space $PSU(2,2|4)/SO(4,1) \times SO(5)$. This paper develops an alternative approach in which the Grassmann coordinates provide a nonlinear realization of $PSU(2,2|4)$ based on the quotient space $PSU(2,2|4)/SU(2,2) \times SU(4)$, and the bosonic coordinates are described as a submanifold of $SU(2,2) \times SU(4)$. This formulation is used to construct the superstring world-sheet action in a form in which the $PSU(2,2|4)$ symmetry is manifest and local kappa symmetry can be established. It provides the complete dependence on the Grassmann coordinates in terms of simple analytic expressions. Therefore it is expected to have advantages compared previous approaches, but this remains to be demonstrated.

[1]jhs@theory.caltech.edu

Contents

1 Introduction

The conjectured duality [1] between type IIB superstring theory [2] in a maximally supersymmetric $AdS_5 \times S^5$ background, with N units of self-dual five-form flux, and four-dimensional $\mathcal{N} = 4$ super Yang–Mills theory [3], with a $U(N)$ gauge group, has been studied extensively. This is a precisely defined conjecture, because the $AdS_5 \times S^5$ background is an exact solution of type IIB superstring theory [4]. Most studies have focused on the large-N limit for fixed 't Hooft parameter $\lambda = g_{YM}^2 N$. (See [5] and references therein.) This limit corresponds to the planar approximation to the field theory [6] and the classical (or leading genus) approximation to the string theory. The planar approximation to the field theory is an integrable four-dimensional theory, with an infinite-dimensional Yangian symmetry generated by the superconformal group $PSU(2,2|4)$ and a dual conformal group. Its perturbative expansion parameter is proportional to λ.

The isometry supergroup of the $AdS_5 \times S^5$ solution of type IIB superstring theory is also $PSU(2,2|4)$. Its bosonic subgroup is $SU(4) \times SU(2,2)$, where $SU(4) = Spin(6)$ and $SU(2,2) = Spin(4,2)$. This supergroup has 32 fermionic generators, which we will refer to as supersymmetries. This is the maximum number possible and the same number as the flat-space solution, which corresponds to the large-radius (or large-λ) limit of the $AdS_5 \times S^5$ solution. The string theory, for the background in question, is described by an interacting two-dimensional world-sheet theory, whose perturbative expansion parameter is proportional to $1/\sqrt{\lambda}$. This theory is also integrable.

Even though the planar $\mathcal{N} = 4$ super Yang–Mills theory and the leading-genus $AdS_5 \times S^5$ superstring action are both integrable, both of them are also very challenging to study. Nonetheless, a lot of progress has been achieved, providing convincing evidence in support of the duality, thanks to an enormous effort by many very clever people. The goal of the present work is to derive a new formulation of the superstring world-sheet theory. It will turn out to be equivalent to the previous formulation by Metsaev and Tseytlin [7] and others [8] [9].[2] However, it has some attractive features that might make it more useful.

In recent work the author has studied the bosonic truncation of the world-volume action of a probe D3-brane embedded in this background and made certain conjectures concerning an interpretation of this action that should hold when the fermionic degrees of freedom are incorporated [11][12]. This provided the motivation for developing a convenient formalism for adding the fermions in which all of the symmetries can be easily understood. While that is the motivation, the present work does not require the reader to be familiar with those papers,

[2]For a recent review, see [10].

nor does it depend on the correctness of their conjectures, which have aroused considerable skepticism. This paper, which is about superspace geometry and the superstring action, does not make any bold conjectures, and therefore it should be noncontroversial.

String world-sheet actions have much in common with WZW models for groups, supergroups, cosets, etc. though there a few differences. One difference is that they are invariant under reparametrization of the world-sheet coordinates. One way of implementing this is to couple the sigma model to two-dimensional gravity. If one chooses a conformally flat gauge, the action reduces to the usual two-dimensional Minkowski space form, but it is supplemented by Virasoro constraints. Residual symmetries in this gauge allow further gauge fixing, the main example being light-cone gauge. In addition to the local reparametrization invariance, superstring actions also have local fermionic symmetries, called kappa symmetry. They are rather subtle, and they play a crucial role. One of the goals of this paper is to give a clear explanation of how kappa symmetry is realized.

In constructing chiral sigma models for homogeneous spaces that have an isometry group G, but are not group manifolds, the standard approach is to formulate them as coset theories. Thus, for example, a theory on a sphere S^n is formulated as an $SO(n+1)/SO(n)$ coset theory. The formulas that describe symmetric spaces as $M = G/H$ coset theories are well-known [13][14]. They involve a construction that incorporates global G symmetry and local H symmetry. This is the standard thing to do, and so it is not surprising that this is the approach that was utilized in [7] to construct the superstring world-sheet action for $AdS_5 \times S^5$. In this case the coset in question is $PSU(2,2|4)/SO(4,1) \times SO(5)$. This paper describes an alternative procedure in which the Grassmann coordinates provide a nonlinear realization of $PSU(2,2|4)$ based on the quotient space $PSU(2,2|4)/SU(2,2) \times SU(4)$, and the bosonic coordinates are described as a submanifold of $SU(2,2) \times SU(4)$.

The description of S^2 as a subspace of $SU(2)$ is a very simple analog of the procedure that will be used. (It is relevant to the discussion of $AdS_2 \times S^2$, which is analogous to $AdS_5 \times S^5$.) The group $SU(2)$ consists of 2×2 unitary unimodular matrices, and the group manifold is S^3. An S^2 can be embedded in this group manifold in many different ways. The one that is most relevant for our purposes is the subspace consisting of all symmetric $SU(2)$ matrices. This subspace can be expressed in terms of Pauli matrices in the form $\sigma_2 \vec{\sigma} \cdot \hat{x}$, where $\hat{x}$ is a unit-length 3-vector. The action of a group element g on an element of this sphere, represented by a symmetric matrix g_0, is $g_0 \to g^T g_0 g$. This is a point on the same S^2, since $g^T g_0 g$ is also a symmetric $SU(2)$ matrix. The isometry group of S^2 is actually $SO(3)$, because the group elements g and $-g$ describe the same map. Clearly, a specific subspace of $SU(2)$ has been selected, in a way that does not depend on any arbitrary choices, to

describe S^2. This paper applies a similar procedure to the description of S^5 and AdS_5 as subspaces of $SU(4)$ and $SU(2,2)$, respectively. In particular, the description of S^5 in terms of antisymmetric $SU(4)$ matrices is discussed in detail in Appendix A.

This formulation of the superspace geometry that enters in the construction of the superstring action makes it possible to keep all of the bosonic symmetries manifest throughout the analysis,[3] and many formulas, including the superstring action itself, have manifest $PSU(2,2|4)$ symmetry. Also, the complete dependence on the Grassmann coordinates for all relevant quantities is given by simple tractable analytic expressions. So far, we have just rederived results that have been known for a long time, but the hope is that this reformulation of the world-sheet theory will be helpful for obtaining new results.

2 The bosonic truncation

Before confronting superspace geometry, let us briefly review the bosonic structure of $AdS_5 \times S^5$, which has the isometry $SO(4,2) \times SO(6)$. The generators of $SO(6)$, denoted J^{ab}, where $a, b = 1, 2, \ldots 6$, can be viewed as generators of rotations of $\mathbb{R}^6$ about the origin. They also generate the isometries of a unit-radius S^5 centered about the origin ($\hat{z} \cdot \hat{z} = \sum_1^6 (z^a)^2 = 1$). Similarly, J^{mn} generates isometries of a unit-radius AdS_5 embedded in $\mathbb{R}^{4,2}$ by the equation

$$\hat{y} \cdot \hat{y} = \sum_{m,n=0}^{5} \eta_{mn} y^m y^n = -(y^0)^2 + (y^1)^2 + (y^2)^2 + (y^3)^2 + (y^4)^2 - (y^5)^2 = -1. \tag{1}$$

This equation describes the Poincaré patch of AdS_5, which is all that we are concerned with in this work. The two algebras are distinguished by the choice of indices (a, b, c, d or m, n, p, q).

We prefer to write the unit-radius $AdS_5 \times S^5$ metric in a form in which all of the isometries are manifest. There are various ways to achieve this. One option is

$$ds^2 = d\hat{z} \cdot d\hat{z} + d\hat{y} \cdot d\hat{y}, \tag{2}$$

where $\hat{z}$ and $\hat{y}$ are understood to satisfy the constraints described above. This is the description that will be utilized in most of this paper.

Lie-algebra-valued connection one-forms associated to the $SO(6)$ symmetry of S^5 are easily constructed in terms of the unit six-vector z^a. (We do not display hats to avoid clutter.) The one-form is

$$\Omega_0^{ab} = 2(z^a dz^b - z^b dz^a). \tag{3}$$

[3] A previous attempt to make the $SU(4)$ symmetry manifest is described in [15].

The subscript 0 is used to refer to the bosonic truncation. The normalization is chosen to ensure that this is a flat connection, *i.e.,* its two-form curvature is

$$d\Omega_0^{ab} + \Omega_0^{ac} \wedge \eta_{cd}\, \Omega_0^{db} = 0. \tag{4}$$

This is easily verified using $z^a\, \eta_{ab}\, dz^b = 0$, which is a consequence of $z^2 = z^a\, \eta_{ab}\, z^b = 1$. In the case of $SO(6)$, the metric η is just a 6×6 unit matrix, which we denote I_6. Similarly, there is a Lie-algebra-valued flat connection

$$\tilde{\Omega}_0^{mn} = -2(y^m dy^n - y^n dy^m) \tag{5}$$

associated to the $SO(4,2)$ symmetry of AdS_5. In the $SO(4,2)$ case $\eta = I_{4,2}$, which has diagonal components $(1,1,1,1,-1,-1)$. Recall that for this choice $y^2 = -1$, which is why $\tilde{\Omega}_0^{mn}$ requires an extra minus sign to ensure flatness.

The bosonic truncation of the superstring action can be expressed entirely in terms of the induced world-volume metric,

$$G_{\alpha\beta} = \partial_\alpha \hat{z} \cdot \partial_\beta \hat{z} + \partial_\alpha \hat{y} \cdot \partial_\beta \hat{y}, \tag{6}$$

where it is understood that y and z are functions of the world-sheet coordinates σ^α, $\alpha = 0, 1$. The action is

$$S = -\frac{R^2}{2\pi\alpha'} \int d^2\sigma \sqrt{-G}, \tag{7}$$

where $G = \det G_{\alpha\beta}$ and α' is the usual string theory Regge slope parameter, which (for $\hbar = c = 1$) has dimensions of length squared. R is the radius of both S^5 and AdS_5. A standard rewriting of such a metric involves introducing an auxiliary world-sheet metric field $h_{\alpha\beta}$. Then the action can be recast as

$$S = -\frac{R^2}{4\pi\alpha'} \int d^2\sigma \sqrt{-h} h^{\alpha\beta} G_{\alpha\beta}. \tag{8}$$

This form has a Weyl symmetry given by an arbitrary local rescaling of $h_{\alpha\beta}$. The simplest way to understand the equivalence of the two forms of S is to note that the $h_{\alpha\beta}$ classical equation of motion is solved by $h_{\alpha\beta} = e^{f(\sigma)} G_{\alpha\beta}$, *i.e.,* they are conformally equivalent. The conformal factor cancels out classically. For a critical string theory, without conformal anomaly, it should also cancel quantum mechanically. The bosonic truncation described here is not critical, but the complete theory should be.

When the fermionic degrees of freedom are included, the dual CFT is $\mathcal{N} = 4$ super Yang–Mills theory with a $U(N)$ gauge group. N is related to a five-form flux, which does not appear in the string world-sheet action. (It does appear in the D3-brane action.) The

gauge theory has a dimensionless 't Hooft parameter $\lambda = g_{YM}^2 N$. AdS/CFT duality gives the identification

$$g_s = \frac{g_{YM}^2}{4\pi}, \tag{9}$$

where g_s is the string coupling constant (determined by the vev of the dilaton field). The radius R of the S^5 and the AdS_5 is introduced by replacing the unit-radius metric ds^2 by $R^2 ds^2$. Then, utilizing the AdS/CFT identification

$$R^2 = \alpha' \sqrt{\lambda}, \tag{10}$$

one obtains

$$S = -\frac{\sqrt{\lambda}}{2\pi} \int d^2\sigma \sqrt{-G}. \tag{11}$$

In the large N limit, taken at fixed λ, the CFT is described by the planar approximation, and the string theory is described by the classical approximation, *i.e.,* leading order in the world-sheet genus expansion, which is a cylinder. Even so, the two-dimensional world-sheet theory must be treated as a quantum theory, with a large-λ perturbation expansion in powers of $1/\sqrt{\lambda}$, in order to determine the string spectrum and tree amplitudes. Flat ten-dimensional spacetime is the leading approximation in this expansion. The dual planar CFT, on the other hand, has a small-λ perturbation expansion in powers of λ.

Let us introduce null world-sheet coordinates $\sigma^\pm = \sigma^1 \pm \sigma^0$. It is often convenient to choose a conformally flat gauge. This means using the two diffeomorphism symmetries to set

$$G_{++} = G_{--} = 0. \tag{12}$$

Then the action simplifies to

$$S = -\frac{\sqrt{\lambda}}{2\pi} \int d^2\sigma G_{+-}, \tag{13}$$

which is supplemented by the Virasoro constraints $G_{++} = G_{--} = 0$.[4] In the geometry at hand, we have

$$G_{+-} = \partial_+ \hat{z} \cdot \partial_- \hat{z} + \partial_+ \hat{y} \cdot \partial_- \hat{y}. \tag{14}$$

This can then be varied to give equations of motion. Taking account of the constraints $\hat{z} \cdot \hat{z} = 1$ and $\hat{y} \cdot \hat{y} = -1$, we obtain

$$(\eta^{ab} - z^a z^b)\partial_+\partial_- z_b = 0 \quad \text{and} \quad (\eta^{mn} + y^m y^n)\partial_+\partial_- y_n = 0. \tag{15}$$

[4]When the world-sheet theory is quantized, G_{++} and G_{--} become operators that need to be treated with care. In any case, the bosonic truncation of the world-sheet theory is inconsistent beyond the classical approximation due to a conformal anomaly.

Conservation of the $SO(6)$ and $SO(4,2)$ Noether currents implies that

$$\partial_\alpha(z^a\partial^\alpha z^b - z^b\partial^\alpha z^a) = 0 \quad \text{and} \quad \partial_\alpha(y^m\partial^\alpha y^n - y^n\partial^\alpha y^m) = 0. \tag{16}$$

These 30 equations are consequences of the 10 preceding equations. In fact, they are equivalent to them. Expressed more elegantly, they take the form

$$d \star \Omega_0^{ab} = 0 \quad \text{and} \quad d \star \tilde{\Omega}_0^{mn} = 0. \tag{17}$$

The bosonic connections Ω_0 and $\tilde{\Omega}_0$ are simultaneously conserved and flat when the equations of motion are taken into account. These conditions allow one to construct a one-parameter family of flat connections, whose existence is the key to classical integrability of the world-sheet theory [16]. In the remainder of this manuscript we will add Grassmann coordinates and construct the complete superstring action with $PSU(2,2|4)$ symmetry. Since this will be a "critical" string theory (without conformal anomaly), its integrability is expected to be valid for the quantum theory, *i.e.,* taking full account of the dependence on λ, but only at leading order in the genus expansion.

3 Supersymmetrization

Our goal is to add fermionic (Grassmann) coordinates θ to the metric of the preceding section so as to make it invariant under $PSU(2,2|4)$. In addition to bosonic one-forms Ω^{ab} and $\tilde{\Omega}^{mn}$, whose bosonic truncations are Ω_0^{ab} and $\tilde{\Omega}_0^{mn}$ described in Sect. 2, we also require a fermionic one-form Ψ, which is dual to the fermionic supersymmetry generators of the superalgebra. Ψ and $\Psi^\dagger$ should encode 32 fermionic one-forms, which transform under $SU(4)\times SU(2,2)$ as $(\mathbf{4},\bar{\mathbf{4}}) + (\bar{\mathbf{4}},\mathbf{4})$.

Let us recast the connections Ω and $\tilde{\Omega}$ in spinor notation. The construction for $SU(4)$ requires 4×4 analogs of Pauli matrices, or Dirac matrices, denoted Σ^a, which are described in Appendix A. In the notation described there, we define

$$\Omega^\alpha{}_\beta = \frac{1}{4}(\Sigma_{ab})^\alpha{}_\beta\Omega^{ab}. \tag{18}$$

Also in the notation described in Appendix A, there is an identical-looking formula for $SU(2,2)$,

$$\tilde{\Omega}^\mu{}_\nu = \frac{1}{4}(\Sigma_{mn})^\mu{}_\nu\tilde{\Omega}^{mn}. \tag{19}$$

Infinitesimal parameters of $SU(4)$ and $SU(2,2)$ transformations are described in spinor notation by matrices $\omega^\alpha{}_\beta$ and $\tilde{\omega}^\mu{}_\nu$ in an analogous manner.

The fermionic one-form, transforming as $(\mathbf{4}, \bar{\mathbf{4}})$, is written $\Psi^\alpha{}_\mu$. Its hermitian conjugate, which transforms as $(\bar{\mathbf{4}}, \mathbf{4})$ is written $(\Psi^\dagger)^\mu{}_\alpha$. Spinor indices can be lowered or contracted using the 4×4 invariant tensors $\eta_{\alpha\bar{\beta}}$ and $\eta_{\mu\bar{\nu}}$. $\eta_{\alpha\bar{\beta}}$ is just the unit matrix I_4, and $\eta_{\mu\bar{\nu}}$ is the $SU(2,2)$ metric $I_{2,2}$. Thus, for example, $\Psi^{\alpha\bar{\mu}} = \Psi^\alpha{}_\nu \eta^{\nu\bar{\mu}}$. By always using matrices with unbarred indices we avoid the need to ever display η matrices explicitly. The price one pays for this is that expressions that are called adjoints, such as $\Psi^\dagger$, are not conventional adjoints, since they contain additional η factors. However, this "adjoint" is still an involution, since the square of an η is a unit matrix. In this notation, it makes sense to call the matrix $\tilde{\Omega}$ antihermitian despite the indefinite signature of $SU(2,2)$.

3.1 Supermatrices

Since it is convenient to represent supergroups using supermatrices, let us review a few basic facts and our conventions. There are various conventions in the literature, and we shall introduce yet another one. We write an 8×8 supermatrix in terms of 4×4 blocks as follows

$$M = \left(\begin{array}{cc} a & \zeta b \\ \zeta c & d \end{array} \right), \tag{20}$$

where a and d are Grassmann even and b and c are Grassmann odd. a is the $SU(4)$ block and d is the $SU(2,2)$ block. This formula contains the phase

$$\zeta = e^{-i\pi/4}, \tag{21}$$

which satisfies $\zeta^2 = -i$. By introducing factors of ζ in this way various formulas have a more symmetrical appearance than is the case for other conventions.

The *superadjoint* is defined by

$$M^\dagger = \left(\begin{array}{cc} a^\dagger & -\zeta c^\dagger \\ -\zeta b^\dagger & d^\dagger \end{array} \right). \tag{22}$$

This definition, which reduces to the usual one for the even blocks, is chosen to ensure the identity $(M_1 M_2)^\dagger = M_2^\dagger M_1^\dagger$. By definition, a unitary supermatrix satisfies $MM^\dagger = I$ and an antihermitian supermatrix satisfies $M + M^\dagger = 0$. Similarly, the *supertranspose* is defined by

$$M^T = \left(\begin{array}{cc} a^T & -i\zeta c^T \\ -i\zeta b^T & d^T \end{array} \right). \tag{23}$$

This satisfies $(M_1 M_2)^T = M_2^T M_1^T$. However, it has the somewhat surprising property

$$(M^T)^T = \left(\begin{array}{cc} a & -\zeta b \\ -\zeta c & d \end{array} \right), \tag{24}$$

which makes the supertranspose a $\mathbb{Z}_4$ transformation [17]. Note that in verifying these formulas, it is important to use the rules $(bc)^\dagger = -c^\dagger b^\dagger$ and $(bc)^T = -c^T b^T$ for Grassmann odd matrices. It will also be useful to define the inverse transpose, which has the same properties as the transpose,

$$M^{\overline{T}} = \begin{pmatrix} a^T & i\zeta c^T \\ i\zeta b^T & d^T \end{pmatrix}. \tag{25}$$

Which is which is a matter of convention.

The *supertrace* is defined by

$$\mathrm{str}M = \mathrm{tr}\, a - \mathrm{tr}\, d. \tag{26}$$

One virtue of this definition is that the familiar identity $\mathrm{tr}(a_1 a_2) = \mathrm{tr}(a_2 a_1)$ generalizes to

$$\mathrm{str}(M_1 M_2) = \mathrm{str}(M_2 M_1). \tag{27}$$

Another virtue is that $\mathrm{str}M^T = \mathrm{str}M^{\overline{T}} = \mathrm{str}M$.

Our main concern in this work is the supergroup $PSU(2,2|4)$. The corresponding superalgebra is best described in terms of matrices belonging to the superalgebra $\mathfrak{su}(2,2|4)$. This algebra consists of antihermitian supermatrices with vanishing supertrace. (It is implicit here that one takes appropriate account of the indefinite signature of $\mathfrak{su}(2,2)$.) Given this algebra, one defines the $\mathfrak{psu}(2,2|4)$ algebra to consist of $\mathfrak{su}(2,2|4)$ matrices modded out by the equivalence relation $M \sim M + i\lambda I$, where I denotes the unit supermatrix. The factor of i is shown because λ is assumed to be real and M is supposed to be antihermitian.

3.2 Nonlinear realization of the superalgebra

Superspace is described by the bosonic spacetime coordinates y^m and z^a, satisfying $z^2 = 1$ and $y^2 = -1$, introduced in Sect. 2, and Grassmann coordinates $\theta^\alpha{}_\mu$. The θ coordinates are 16 complex Grassmann numbers that transform under $SU(4) \times SU(2,2)$ as $(\mathbf{4}, \bar{\mathbf{4}})$, like the one-form Ψ discussed above. It will be extremely helpful to think of θ as a 4×4 matrix rather than as a 16-component spinor. The two points of view are equivalent, of course, but matrix notation will lead to much more elegant formulas. If all matrix multiplications were done from one side, a tensor product notation (like that in Appendix C) would be required. Using matrix notation, we will obtain simple analytic expressions describing the full θ dependence of all quantities that are required to formulate the superstring action.

One clue to understanding the $PSU(2,2|4)$ symmetry of the $AdS_5 \times S^5$ geometry is its relationship to the super-Poincaré symmetry algebra of flat ten-dimensional superspace, which corresponds to the large-radius limit. The large-radius limit preserves all 32 fermionic

symmetries, but it only accounts for 30 of the 55 bosonic symmetries of the Poincaré algebra in 10 dimensions. The 25 rotations and Lorentz transformations that relate the $\mathbb{R}^{4,1}$ piece of the geometry that descends from AdS_5 to the $\mathbb{R}^5$ piece that descends from S^5 are additional "accidental" symmetries of the limit.

A little emphasized feature of the superspace description of the flat-space geometry is that the entire super-Poincaré algebra closes on the Grassmann coordinates θ. A possible reason for this lack of emphasis may be that the ten spacetime translations act trivially, *i.e.,* they leave θ invariant. We will demonstrate here that the entire $\mathfrak{psu}(2,2|4)$ superalgebra closes on the fermionic coordinates $\theta^\alpha{}_\mu$ even though the radius is finite. In this case all of the symmetries transform θ nontrivially, and none of the transformations of θ give rise to expressions involving the y or z coordinates. This means that the Grassmann coordinates provide a nonlinear realization of the superalgebra. Conceptually, this is similar to the way the supersymmetry of a field theory in flat spacetime is realized nonlinearly on a spinor field (the Goldstino) [20]. In fact, the algebra for the two problems is quite similar. The nonlinear Lagrangian for the Goldstino field was generalized to anti de Sitter space in [21]. However, that work is not directly relevant, since the goal of the present work is to describe world-sheet fields and not ten-dimensional target-space fields. The latter may deserve further consideration in the future.

The infinitesimal bosonic symmetry transformations of θ are relatively trivial; they are "manifest" in the sense that they are determined by the types of spinor indices that appear. In matrix notation,

$$\delta\theta^\alpha{}_\mu = (\omega\theta - \theta\tilde{\omega})^\alpha{}_\mu. \tag{28}$$

The infinitesimal parameters $\omega^\alpha{}_\beta$ and $\tilde{\omega}^\mu{}_\nu$ take values in the $\mathfrak{su}(4)$ and $\mathfrak{su}(2,2)$ Lie algebras, respectively. Thus, they are anti-hermitian (in the sense discussed earlier) and traceless.

Let us now consider an infinitesimal supersymmetry transformation of θ. In the case of flat space this is just $\delta\theta = \varepsilon$, where ε is an infinitesimal constant matrix of complex Grassmann parameters. In the case of unit radius it is a bit more interesting:[5]

$$\delta\theta^\alpha{}_\mu = \varepsilon^\alpha{}_\mu + i(\theta\varepsilon^\dagger\theta)^\alpha{}_\mu. \tag{29}$$

The hermitian conjugate equation is then

$$\delta(\theta^\dagger)^\mu{}_\alpha = (\varepsilon^\dagger)^\mu{}_\alpha + i(\theta^\dagger\varepsilon\theta^\dagger)^\mu{}_\alpha. \tag{30}$$

We have displayed the spinor indices, but the more compact formulas $\delta\theta = \varepsilon + i\theta\varepsilon^\dagger\theta$ and

[5] This formula has appeared previously in [18][19].

$\delta\theta^\dagger = \varepsilon^\dagger + i\theta^\dagger\varepsilon\theta^\dagger$ are completely unambiguous. In our notation, the quantities

$$u = i\theta\theta^\dagger \quad \text{and} \quad \tilde{u} = i\theta^\dagger\theta \tag{31}$$

are both hermitian.

In our conventions all coordinates (θ, y, z) are dimensionless, since they pertain to unit radius ($R = 1$). If we were to give them the usual dimensions, by absorbing appropriate powers of R, then the second term in $\delta\theta = \varepsilon + i\theta\varepsilon^\dagger\theta$ would contain a coefficient $1/R$. This makes it clear that this term vanishes in the large-radius limit.

It is a beautiful exercise to compute the commutator of two of these supersymmetry transformations,

$$\begin{aligned}
[\delta_1, \delta_2]\theta &= \delta_1(\varepsilon_2 + i\theta\varepsilon_2^\dagger\theta) - (1 \leftrightarrow 2) \\
&= i(\varepsilon_1 + i\theta\varepsilon_1^\dagger\theta)\varepsilon_2^\dagger\theta + i\theta\varepsilon_2^\dagger(\varepsilon_1 + i\theta\varepsilon_1^\dagger\theta) - (1 \leftrightarrow 2) \\
&= \omega_{12}\theta - \theta\tilde{\omega}_{12},
\end{aligned} \tag{32}$$

where ω_{12} and $\tilde{\omega}_{12}$ are

$$(\omega_{12})^\alpha{}_\beta = i(\varepsilon_1\varepsilon_2^\dagger - \varepsilon_2\varepsilon_1^\dagger)^\alpha{}_\beta - \text{trace}, \tag{33}$$

$$(\tilde{\omega}_{12})^\mu{}_\nu = i(\varepsilon_1^\dagger\varepsilon_2 - \varepsilon_2^\dagger\varepsilon_1)^\mu{}_\nu - \text{trace}. \tag{34}$$

These are antihermitian, as required, since $(\varepsilon_1\varepsilon_2^\dagger)^\dagger = -\varepsilon_2\varepsilon_1^\dagger$. Traces are subtracted in order that they are Lie-algebra valued. This is possible due to the fact that the two trace terms give canceling contributions to Eq. (32). This commutator is exactly what the superalgebra requires it to be, demonstrating that $\mathfrak{psu}(2,2|4)$ is nonlinearly realized entirely in terms of the Grassmann coordinates.

The transformation rule in Eq. (29) is not a unique choice. The nonuniqueness corresponds to the possibility of redefining θ by introducing $\theta' = \theta + ic_1\theta\theta^\dagger\theta + \ldots$. Then, the transformation rule would be modified accordingly. One could even incorporate y and z in a redefinition, which would be truly perverse. The choice that we have made is clearly the simplest and most natural one, so it will be used in the remainder of this work.

It is possible to construct elements of the supergroup, represented by unitary supermatrices, which are constructed entirely out of the Grassmann coordinates. For this purpose, let us consider the supermatrix[6]

$$\Gamma = I(\theta)\hat{f}^{-1} = \hat{f}^{-1}I(\theta), \tag{35}$$

[6] This description was suggested by W. Siegel, who brought his related work to our attention [22] [23] [24].

where

$$I(\theta) = \begin{pmatrix} I & \zeta\theta \\ \zeta\theta^\dagger & I \end{pmatrix} \tag{36}$$

and

$$\hat{f} = \begin{pmatrix} f & 0 \\ 0 & \tilde{f} \end{pmatrix}. \tag{37}$$

In this formula f denotes a real analytic function of the hermitian matrix $u = i\theta\theta^\dagger$ and $\tilde{f}$ denotes the same function with argument $\tilde{u} = i\theta^\dagger\theta$. These functions are actually polynomials of degree 16 or less, since higher powers necessarily vanish. The fact that $[I(\theta), \hat{f}] = 0$ is a consequence of the identities

$$f\theta = \theta\tilde{f} \quad \text{and} \quad \theta^\dagger f = \tilde{f}\theta^\dagger. \tag{38}$$

The choice of the function f is determined by requiring that Γ is superunitary, *i.e.,* $\Gamma^\dagger\Gamma = I$. Using the definition of the superadjoint given in Eq. (22),

$$\Gamma^\dagger = I(-\theta)\hat{f}^{-1} = \hat{f}^{-1}I(-\theta). \tag{39}$$

The requirement $\Gamma^\dagger\Gamma = I$ then becomes

$$\hat{f}^2 = I(-\theta)I(\theta) = \begin{pmatrix} I + u & 0 \\ 0 & I + \tilde{u} \end{pmatrix}. \tag{40}$$

Therefore the correct choices for f and $\tilde{f}$ are the hermitian matrices

$$f = \sqrt{I+u} = I + \frac{1}{2}u + \ldots \quad \text{and} \quad \tilde{f} = \sqrt{I+\tilde{u}} = I + \frac{1}{2}\tilde{u} + \ldots. \tag{41}$$

For this choice Γ is an element of the supergroup $SU(2,2|4)$.

3.3 Grassmann-valued connections

Various one-forms that can be regarded as connections associated to the superalgebra will arise in the course of this work. Here we utilize the nonlinear realization that we just found to construct ones that only involve the Grassmann coordinates. The y and z coordinates will need to be incorporated later, and then additional connections will be defined.

Consider the super-Lie-algebra-valued one-form

$$\mathcal{A} = \Gamma^{-1}d\Gamma = \begin{pmatrix} K & \zeta\Psi \\ \zeta\Psi^\dagger & \tilde{K} \end{pmatrix}. \tag{42}$$

This supermatrix is super-antihermitian, as required. (The matrices required to take account of the indefinite signature of $\mathfrak{su}(2,2)$ are implicit, as discussed earlier.) Explicit calculation gives

$$K = -df f^{-1} + i\theta\Psi^\dagger = f^{-1}df - i\Psi\theta^\dagger, \tag{43}$$

and

$$\tilde{K} = -d\tilde{f}\tilde{f}^{-1} + i\theta^\dagger\Psi = \tilde{f}^{-1}d\tilde{f} - i\Psi^\dagger\theta, \tag{44}$$

where

$$\Psi = f^{-1}d\theta\tilde{f}^{-1} \quad \text{and} \quad \Psi^\dagger = \tilde{f}^{-1}d\theta^\dagger f^{-1}. \tag{45}$$

We prefer to not subtract the trace parts of K and $\tilde{K}$, which would be required to make them elements of $\mathfrak{su}(4)$ and $\mathfrak{su}(2,2)$, respectively. Since we define $\mathfrak{psu}(2,2|4)$ as a quotient space of $\mathfrak{su}(2,2|4)$, it is sufficient for our purposes that $\mathrm{tr}K = \mathrm{tr}\tilde{K}$, which implies that $\mathrm{str}\mathcal{A} = 0$. This ensures that the traces could be removed, as in the case of $\omega_{12}\theta - \theta\tilde{\omega}_{12}$, which was discussed earlier.

The fact that $\mathcal{A}$ is "pure gauge" implies that it is a flat connection, *i.e.,*

$$d\mathcal{A} + \mathcal{A} \wedge \mathcal{A} = 0. \tag{46}$$

In terms of 4×4 blocks the zero-curvature equations are

$$dK + K \wedge K - i\Psi \wedge \Psi^\dagger = 0, \quad d\tilde{K} + \tilde{K} \wedge \tilde{K} - i\Psi^\dagger \wedge \Psi = 0, \tag{47}$$

$$d\Psi + K \wedge \Psi + \Psi \wedge \tilde{K} = 0, \quad d\Psi^\dagger + \tilde{K} \wedge \Psi^\dagger + \Psi^\dagger \wedge K = 0. \tag{48}$$

These equations have the same structure as the Maurer–Cartan equations of the superalgebra.

Under an arbitrary variation $\delta\Gamma$, the variation of $\mathcal{A} = \Gamma^{-1}d\Gamma$ is

$$\delta\mathcal{A} = d(\Gamma^{-1}\delta\Gamma) + [\mathcal{A}, \Gamma^{-1}\delta\Gamma]. \tag{49}$$

The supermatrix Γ depends only on θ. Therefore to determine how $\mathcal{A}$ varies under an arbitrary variation of θ, we need to know the variation $\Gamma^{-1}\delta\Gamma$ for an arbitrary variation of θ. The result is that

$$\Gamma^{-1}\delta\Gamma = \begin{pmatrix} \mathcal{M} & \zeta\rho \\ \zeta\rho^\dagger & \tilde{\mathcal{M}} \end{pmatrix} = \hat{\mathcal{M}} \tag{50}$$

where

$$\mathcal{M} = -\delta f f^{-1} + i\theta\rho^\dagger = f^{-1}\delta f - i\rho\theta^\dagger \tag{51}$$

$$\tilde{\mathcal{M}} = -\delta\tilde{f}\tilde{f}^{-1} + i\theta^\dagger\rho = \tilde{f}^{-1}\delta\tilde{f} - i\rho^\dagger\theta, \tag{52}$$

and

$$\rho = f^{-1}\delta\theta\tilde{f}^{-1}. \tag{53}$$

These formulas for $\Gamma^{-1}\delta\Gamma$ have exactly the same structure as the previous ones for $\mathcal{A} = \Gamma^{-1}d\Gamma$. So no additional computation was required to derive them. They will be useful later when we derive equations of motion.

A special case of these formulas that is of particular interest is when the variation $\delta\theta$ is not arbitrary but rather is an infinitesimal supersymmetry transformation of the form given in Eq. (29). In that case we find

$$\delta_\varepsilon\Gamma = \Gamma\begin{pmatrix} M & 0 \\ 0 & \tilde{M} \end{pmatrix} + \begin{pmatrix} 0 & \zeta\varepsilon \\ \zeta\varepsilon^\dagger & 0 \end{pmatrix}\Gamma, \tag{54}$$

where

$$M = -(\delta_\varepsilon f - if\varepsilon\theta^\dagger)f^{-1} = f^{-1}(\delta_\varepsilon f - i\theta\varepsilon^\dagger f) \tag{55}$$

and

$$\tilde{M} = -(\delta_\varepsilon\tilde{f} - i\tilde{f}\varepsilon^\dagger\theta)\tilde{f}^{-1} = \tilde{f}^{-1}(\delta_\varepsilon\tilde{f} - i\theta^\dagger\varepsilon\tilde{f}). \tag{56}$$

Equation (54) shows that under a supersymmetry transformation Γ is multiplied on the right by a local $\mathfrak{su}(4) \times \mathfrak{su}(2,2)$ transformation and on the left by a global supersymmetry transformation, just as one would expect in a coset construction. This supports interpreting the nonlinear realization of $PSU(2,2|4)$ in terms of θ as a coset construction[7]

$$PSU(2,2|4)/SU(4) \times SU(2,2). \tag{57}$$

The global supersymmetry term, *i.e.,* the second term on the right-hand side of Eq. (54), does not contribute to $\delta_\varepsilon\mathcal{A}$, which takes the form

$$\delta_\varepsilon\mathcal{A} = d(\Gamma^{-1}\delta_\varepsilon\Gamma) + [\mathcal{A}, \Gamma^{-1}\delta_\varepsilon\Gamma] = d\hat{M} + [\mathcal{A}, \hat{M}]. \tag{58}$$

where

$$\hat{M} = \begin{pmatrix} M & 0 \\ 0 & \tilde{M} \end{pmatrix}. \tag{59}$$

In terms of blocks this gives

$$\delta_\varepsilon K = dM + [K, M] \quad \text{and} \quad \delta_\varepsilon\tilde{K} = d\tilde{M} + [\tilde{K}, \tilde{M}], \tag{60}$$

$$\delta_\varepsilon\Psi = \Psi\tilde{M} - M\Psi \quad \text{and} \quad \delta_\varepsilon\Psi^\dagger = \Psi^\dagger M - \tilde{M}\Psi^\dagger. \tag{61}$$

[7] This was pointed out by E. Witten.

3.4 Inclusion of bosonic coordinates

The formulas that have been described in this section so far describe the supermanifold geometry for fixed values of the bosonic coordinates y and z, introduced in Sect. 2, which we sometimes refer to collectively as x. Our goal now is to describe the generalization that also allows the bosonic coordinates to vary. For this purpose, the first step is to recast y and z as 4×4 matrices denoted Y and Z. This is described in detail in Appendix A. The result that is established there is that

$$Y^{\mu\nu} = y^m (\tilde{\Sigma}_m)^{\mu\nu} \quad \text{and} \quad Z^{\alpha\beta} = z^a (\Sigma_a)^{\alpha\beta} \tag{62}$$

are antisymmetric matrices belonging to the groups $SU(2,2)$ and $SU(4)$, respectively. Thus, in our notation, $Y^T = -Y$, $Z^T = -Z$, $Y^{-1} = Y^\dagger$, $Z^{-1} = Z^\dagger$, and $\det Y = \det Z = 1$. These equations are consequences of the relations $y^2 = -1$ and $z^2 = 1$, as well as Clifford-algebra-like formulas for the Σ matrices. Thus, S^5 is described as a specific codimension 10 submanifold of the $SU(4)$ group manifold, and AdS_5 is described as a specific codimension 10 submanifold of the $SU(2,2)$ group manifold.

The supersymmetry transformations of the bosonic coordinates, which are encoded in the antisymmetric matrices Z and Y, are given by induced local $SU(4)$ and $SU(2,2)$ transformations

$$\delta_\varepsilon Z = -(MZ + ZM^T) \quad \text{and} \quad \delta_\varepsilon Y = -(\tilde{M}Y + Y\tilde{M}^T). \tag{63}$$

Defining the supermatrix

$$X = \begin{pmatrix} Z & 0 \\ 0 & Y \end{pmatrix}, \tag{64}$$

these can be combined into the supermatrix equation

$$\delta_\varepsilon X = -(\hat{M}X + X\hat{M}^T). \tag{65}$$

As a check of these formulas, one can verify that the commutator of two such transformations gives the correct infinitesimal $\mathfrak{su}(4)$ and $\mathfrak{su}(2,2)$ transformations. This is achieved in the $\mathfrak{su}(4)$ case by verifying that

$$[M_2, M_1] + \delta_2 M_1 - \delta_1 M_2 = \omega_{12}, \tag{66}$$

where $M_i = M(\varepsilon_i)$, $\delta_i = \delta_{\varepsilon_i}$, and ω_{12} is given in Eq. (33). The $\delta_\varepsilon Y$ equation is established in the same way.

3.5 Majorana–Weyl matrices and Maurer–Cartan equations

In the flat spacetime limit, a fermionic matrix such as θ corresponds to a complex Weyl spinor, which (in the notation of [25]) satisfies an equation of the form $\Gamma_{11}\theta = \theta$. This

spinor describes a reducible representation of the $\mathcal{N} = 2B$, $D = 10$ super-Poincaré group, and so it can be decomposed into a pair of Majorana–Weyl spinors $\theta = \theta_1 + i\theta_2$. In a Majorana representation of the Dirac algebra the MW spinors θ_1 and θ_2 each contain 16 real components. In the case of $PSU(2,2|4)$ the group theory is different. The relevant representation of $SU(2,2) \times SU(4)$ is still reducible, $(\mathbf{4}, \bar{\mathbf{4}}) + (\bar{\mathbf{4}}, \mathbf{4})$, but it does not make group-theoretic sense to extract the real and imaginary parts by adding and subtracting these two pieces. Fortunately, there is a construction that is group theoretically sensible and connects smoothly with the flat-space limit.

The transformations given previously imply that Ψ and

$$\Psi' = Z\Psi^{\star}Y^{-1} \tag{67}$$

transform in the same way under all $PSU(2,2|4)$ transformations. To understand this definition one should follow the indices. The complex conjugate is $(\Psi^{\alpha}{}_{\mu})^{\star} = (\Psi^{\star})^{\bar{\alpha}}{}_{\bar{\mu}}$, but as usual we convert to unbarred indices, $(\Psi^{\star})_{\alpha}{}^{\mu}$, using η matrices, *i.e.,* $\Psi^{\star} \to \eta\Psi^{\star}\eta$. Then $(\Psi')^{\alpha}{}_{\mu} = Z^{\alpha\beta}(\Psi^{\star})_{\beta}{}^{\nu}(Y^{-1})_{\nu\mu}$. Therefore Ψ and Ψ' transform in the same way, and it makes group-theoretic sense to define

$$\Psi_1 = \frac{1}{2}(\Psi + \Psi') \quad \text{and} \quad \Psi_2 = \frac{1}{2i}(\Psi - \Psi'). \tag{68}$$

Then $\Psi = \Psi_1 + i\Psi_2$ and $\Psi' = \Psi_1 - i\Psi_2$. We will refer to Ψ_1 and Ψ_2 as *Majorana–Weyl matrices*. A MW matrix, such as Ψ_1, satisfies the "reality" identity

$$\Psi_1 = Z\Psi_1^{\star}Y^{-1} \quad \text{or} \quad \Psi_1^{\dagger} = Y\Psi_1^{T}Z^{-1}. \tag{69}$$

What we have here is a generalization of complex conjugation given by

$$\rho \to \mu(\rho) = \rho' = Z\rho^{\star}Y^{-1}, \tag{70}$$

where $\rho^{\alpha}{}_{\mu}$ is an arbitrary fermionic matrix (not necessarily a one-form) that transforms under $SU(2,2) \times SU(4)$ transformations like Ψ or θ. Using the antisymmetry and unitarity of Y and Z it is easy to verify that μ is an involution, like complex conjugation, *i.e.,* $\mu \circ \mu = I$, where I is the identity operator. Therefore,

$$\mu_{\pm} = \frac{1}{2}(I \pm \mu) \tag{71}$$

are a pair of orthogonal projection operators that separate ρ into two pieces, $\rho = \rho_1 + i\rho_2$. In the flat-space limit ρ_1 and ρ_2 correspond to conventional MW spinors.

There are two possible definitions of the covariant exterior derivative of X, the supermatrix that represents the bosonic coordinates y and z. They are

$$D_+X = dX + \mathcal{A}X + X\mathcal{A}^T \quad \text{and} \quad D_-X = dX + \mathcal{A}X + X\mathcal{A}^{\overline{T}}. \tag{72}$$

Here, $\overline{T}$ denotes the inverse transpose, which acts on the odd blocks with the opposite sign from the transpose T. It may be surprising that D_+X and D_-X have nonvanishing odd blocks, even though X does not. There is no inconsistency, and these are definitely the most convenient and natural definitions. In particular, $D_\pm X$ transform under a supersymmetry transformation in the same way as X, namely

$$\delta_\varepsilon D_\pm X = -(\hat{M} D_\pm X + D_\pm X \hat{M}^T). \tag{73}$$

Note that $\hat{M}^T = \hat{M}^{\overline{T}}$, since $\hat{M}$ only has even blocks. The antisymmetry of X gives rise to the relation $(D_-X)^T = -D_+X$.

Given these definitions, we can define a pair of connections

$$A_+ = -D_+XX^{-1} = -dXX^{-1} - \mathcal{A} - X\mathcal{A}^T X^{-1} \tag{74}$$

and

$$A_- = -D_-XX^{-1} = -dXX^{-1} - \mathcal{A} - X\mathcal{A}^{\overline{T}} X^{-1}. \tag{75}$$

Inserting the definition of $\mathcal{A}$ and remembering the factors of $\pm i$ in the odd blocks of $\mathcal{A}^T$ and $\mathcal{A}^{\overline{T}}$, these connections can be written in the form

$$A_\pm = A_1 + A_2 \pm iA_3, \tag{76}$$

where

$$A_1 = \begin{pmatrix} \Omega & 0 \\ 0 & \tilde{\Omega} \end{pmatrix}, \quad A_2 = -\begin{pmatrix} 0 & \zeta\Psi \\ \zeta\Psi^\dagger & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & \zeta\Psi' \\ \zeta\Psi'^\dagger & 0 \end{pmatrix}. \tag{77}$$

Also,

$$\Omega = ZdZ^{-1} - K - ZK^TZ^{-1} \quad \text{and} \quad \tilde{\Omega} = YdY^{-1} - \tilde{K} - Y\tilde{K}^TY^{-1}. \tag{78}$$

Each of the three superconnections A_i is super antihermitian, and under a supersymmetry transformation

$$\delta_\varepsilon A_i = [A_i, \hat{M}] \quad i = 1, 2, 3. \tag{79}$$

Using the definition of the supertranspose in Eq. (23), these supermatrices have the important properties

$$XA_1^TX^{-1} = A_1, \quad XA_2^TX^{-1} = iA_3, \quad XA_3^TX^{-1} = iA_2. \tag{80}$$

and therefore

$$XA_-^T X^{-1} = A_+ \quad \text{and} \quad XA_+^{\overline{T}} X^{-1} = A_-. \tag{81}$$

Another useful set of one-form supermatrices is

$$J_i = \Gamma A_i \Gamma^{-1}, \tag{82}$$

where Γ is defined in Eq. (35). For example,

$$J_1 = \begin{pmatrix} f^{-1}(\Omega + i\theta\tilde{\Omega}\theta^\dagger)f^{-1} & \zeta f^{-1}(\Omega\theta - \theta\tilde{\Omega})\tilde{f}^{-1} \\ \zeta\tilde{f}^{-1}(\tilde{\Omega}\theta^\dagger - \theta^\dagger\Omega)f^{-1} & \tilde{f}^{-1}(\tilde{\Omega} + i\theta^\dagger\Omega\theta)\tilde{f}^{-1} \end{pmatrix}. \tag{83}$$

Note that

$$DA_i = dA_i + \mathcal{A} \wedge A_i + A_i \wedge \mathcal{A} = \Gamma^{-1}(dJ_i)\Gamma. \tag{84}$$

Since A_i is antihermitian, the definition of DA_i does not involve transposes, and it is unambiguous. Utilizing Eq. (54), one can show that the transformation of these supermatrices under arbitrary infinitesimal $\mathfrak{psu}(2,2|4)$ transformations is given by [8]

$$\delta_\Lambda J_i = [\Lambda, J_i], \tag{85}$$

where the various infinitesimal parameters are combined in the supermatrix

$$\Lambda = \begin{pmatrix} \omega & \zeta\varepsilon \\ \zeta\varepsilon^\dagger & \tilde{\omega} \end{pmatrix}. \tag{86}$$

The local generalization of these global symmetry transformation rules are used in Appendix B to derive the $\mathfrak{psu}(2,2|4)$ Noether currents of the superstring.

The one-forms $J_\pm = J_1 + J_2 \pm iJ_3$ can be recast in the form

$$J_+ = B_+ dB_+^{-1} \quad \text{and} \quad J_- = B_- dB_-^{-1}, \tag{87}$$

where

$$B_+ = \Gamma X \Gamma^T \quad \text{and} \quad B_- = \Gamma X \Gamma^{\overline{T}}. \tag{88}$$

These relations imply that J_+ and J_- are flat connections

$$dJ_+ + J_+ \wedge J_+ = 0 \quad \text{and} \quad dJ_- + J_- \wedge J_- = 0. \tag{89}$$

It is straightforward to verify that $DA_2 = -2A_2 \wedge A_2$, which implies that $2J_2$ is also a flat connection. The three flatness conditions imply the Maurer–Cartan (MC) equations

$$dJ_1 = -J_1 \wedge J_1 + J_2 \wedge J_2 + J_3 \wedge J_3 - J_1 \wedge J_2 - J_2 \wedge J_1, \tag{90}$$

$$dJ_2 = -2J_2 \wedge J_2, \tag{91}$$

$$dJ_3 = -(J_1 + J_2) \wedge J_3 - J_3 \wedge (J_1 + J_2). \tag{92}$$

[8] Since Λ and $\Lambda + icI$, where I is a unit supermatrix, give the same transformation, the formula only depends on the equivalence class of $\mathfrak{su}(2,2|4)$ matrices that define a $\mathfrak{psu}(2,2|4)$ element.

3.6 Invariant differential forms

Let us consider the construction of differential forms that are closed and invariant under the entire supergroup. Such differential forms of degree $p + 2$ are required to construct the Wess–Zumino (WZ) terms of p-brane actions. Type IIB superstring theory has an infinite $SL(2, \mathbb{Z})$ multiplet of (p, q) strings, but we are primarily interested in the fundamental $(1, 0)$ superstring here. Its WZ term is determined by a three-form. Similarly, the D3-brane world-volume action contains a WZ term determined by a closed and invariant self-dual five-form.

Consider $T_n = \mathrm{str}(J \wedge J \ldots \wedge J)$, the supertrace of an n-fold wedge product of a one-form supermatrix J. This vanishes for n even because the cyclic identity of the supertrace, $\mathrm{str}(AB) = \mathrm{str}(BA)$, acquires an additional minus sign, *i.e.*, $\mathrm{str}(A \wedge B) = -\mathrm{str}(B \wedge A)$, if A and B are supermatrices of differential forms of odd degree. Now suppose that n is odd, so that T_n can be nonzero, and that J is a flat connection ($dJ = -J \wedge J$). In this case the exterior derivative $d\,T_n$ is proportional to T_{n+1}, which is equal to zero. Therefore T_n is closed.

Let us now utilize this logic to construct a closed three-form based on the flat connections that we have found. The simple choice $\mathrm{str}(J_2 \wedge J_2 \wedge J_2) = \mathrm{str}(A_2 \wedge A_2 \wedge A_2)$ is closed, but it is also zero, since the product of the three A_2 factors has vanishing blocks on the diagonal. Therefore, let us consider instead

$$T_3 = \mathrm{str}(J_+ \wedge J_+ \wedge J_+). \tag{93}$$

This is complex, and therefore it appears to encode two real three-forms that are $PSU(2, 2|4)$ invariant and closed. However, Eqs. (81) and (82) imply that $T_3 = -\mathrm{str}(J_- \wedge J_- \wedge J_-)$, and therefore the real part of T_3 vanishes.

Now let us substitute $J_1 + J_2 + iJ_3$ for J_+. Doing this, and only keeping those terms that can give a nonzero contribution to the supertrace, leaves $T_3 = 3iT_F$, where

$$T_F = \mathrm{str}(J_1 \wedge [J_2 \wedge J_3 + J_3 \wedge J_2]). \tag{94}$$

The notation is meant to indicate that T_F enters in the construction of the WZ term of the fundamental string. The closed three-form T_F is exact, since

$$T_F = d\,\mathrm{str}(J_2 \wedge J_3). \tag{95}$$

Therefore the WZ term of the fundamental superstring world-sheet action is proportional to $\int \mathrm{str}(J_2 \wedge J_3)$.

The WZ term derived above can be obtained more directly if one anticipates that the three form is exact. Consider all invariant two-forms of the type $T_{ij} = \mathrm{str}(J_i \wedge J_j)$. Since J_i is a one-form, cyclic permutation gives $T_{ij} = -T_{ji}$. Furthermore, $T_{13} = T_{23} = 0$, because the expressions inside these supertraces contain no diagonal blocks when reexpressed in terms of A's. Thus, up to normalization, the only nonzero invariant two-form of this type is T_{23}, which is the one required to construct the WZ term for the fundamental superstring.

A self-dual five-form, which is closed and $PSU(2,2|4)$ invariant, also plays an important role in type IIB superstring theory in an $AdS_5 \times S^5$ background. It has a nonzero bosonic truncation in contrast to the three-form described above. Its bosonic part is proportional to the sum (or difference) of the volume form of AdS_5 and the volume form of S^5, and therefore it is not exact. The supersymmetric completion of this five-form is proportional to

$$T_5 = \mathrm{str}(J_+ \wedge J_+ \wedge J_+ \wedge J_+ \wedge J_+) = \mathrm{str}(J_- \wedge J_- \wedge J_- \wedge J_- \wedge J_-). \tag{96}$$

This five-form determines the WZ term for the D3-brane world-volume action in the $AdS_5 \times S^5$ background. The complete construction of the D3-brane action will be described elsewhere.

4 The superstring world-sheet theory

4.1 The action

The world-volume actions of supersymmetric probe branes, including the fundamental superstring, are written as a sum of two terms. The first term, which we denote S_1, is of the Nambu–Goto/Volkov–Akulov type.[9] The second term, which we denote S_2, is of the Wess–Zumino (WZ) or Chern–Simons type. Each of these terms is required to have local reparametrization invariance. In the case of S_1, this symmetry can be implemented by introducing a world-sheet metric, as described in Sect. 2. The S_2 term, on the other hand, is independent of the world-sheet metric. The target superspace isometry, which in the present case is $PSU(2,2|4)$, is realized as a global symmetry of S_1 and S_2 separately. Furthermore, there should be a local fermionic symmetry, called kappa symmetry. Kappa symmetry implies that half of the target-space Grassmann coordinates θ are redundant gauge degrees of freedom of the world-volume theory that can be eliminated by a suitable gauge choice. Unlike all of the other symmetries, kappa symmetry is not a symmetry of S_1 and S_2 separately.

[9] Nambu–Goto refers to a pull-back of the target-space metric to the brane. Volkov–Akulov refers to the appearance of Goldstino fields in theories with spontaneously broken supersymmetries – conformal supersymmetries in the present case. In the case of D-branes, the S_1 term also contains a $U(1)$ field strength and is usually said to be of the Dirac–Born–Infeld (DBI) type.

Rather, it requires a conspiracy between them. Given S_1, a specific S_2, unique up to sign, is required by kappa symmetry. In the case of flat ten-dimensional spacetime, the superstring action $S = S_1 + S_2$ turns out to be a free theory, a fact that can be made manifest in light-cone gauge [26]. In this gauge the exact spectrum of the string is easily determined. In the case of an $AdS_5 \times S^5$ background, the superstring world-sheet theory is not a free theory, but it is an integrable theory [27], as we will discuss later.

The construction of S_1 works exactly as explained for the bosonic truncation in Sect. 2. The only change is that now $G_{\alpha\beta}$ is determined by a supersymmetrized target-space metric. The correct choice turns out to be

$$G_{\alpha\beta} = -\frac{1}{4}\mathrm{str}(J_{1\alpha}J_{1\beta}) = -\frac{1}{4}\mathrm{str}(A_{1\alpha}A_{1\beta}) = -\frac{1}{4}\left(\mathrm{tr}(\Omega_\alpha\Omega_\beta) - \mathrm{tr}(\tilde{\Omega}_\alpha\tilde{\Omega}_\beta)\right). \tag{97}$$

The normalization is chosen to give the correct bosonic truncation. As explained in Sect. 2,

$$S_1 = -\frac{\sqrt{\lambda}}{4\pi}\int d^2\sigma\sqrt{-h}h^{\alpha\beta}G_{\alpha\beta}, \tag{98}$$

which is classically equivalent to

$$S_1 = -\frac{\sqrt{\lambda}}{2\pi}\int d^2\sigma\sqrt{-\det G_{\alpha\beta}}. \tag{99}$$

The second term in the superstring world-sheet action, denoted S_2, should also be invariant under the entire $PSU(2,2|4)$ supergroup. Furthermore, as discussed in Sect. 3.6, it must be proportional to $\int \mathrm{str}(J_2 \wedge J_3)$. Its normalization should be chosen such that $S = S_1 + S_2$ has local kappa symmetry. This symmetry implies that half of the θ coordinates are gauge degrees of freedom of the world-sheet theory. Altogether, the superstring action is

$$S = \frac{\sqrt{\lambda}}{16\pi}\int d^2\sigma\left(\sqrt{-h}h^{\alpha\beta}\mathrm{str}(J_{1\alpha}J_{1\beta}) + k\,\varepsilon^{\alpha\beta}\mathrm{str}(J_{2\alpha}J_{3\beta})\right), \tag{100}$$

where the coefficient k will be determined by requiring kappa symmetry, though its sign is a matter of convention. For any k this action has manifest global $PSU(2,2|4)$ symmetry, since for an arbitrary infinitesimal transformation by a constant amount Λ, $\delta_\Lambda J_i = [\Lambda, J_i]$. We will find that kappa symmetry requires $k = \pm 2$ and choose the plus sign. Then the action can be rewritten in the form

$$S = \frac{\sqrt{\lambda}}{16\pi}\int \left(\mathrm{str}(J_1 \wedge \star J_1) + 2\,\mathrm{str}(J_2 \wedge J_3)\right). \tag{101}$$

The Hodge dual is defined here using the metric $h_{\alpha\beta}$, though (at leading order in the world sheet genus expansion) it is equivalent to replace $h_{\alpha\beta}$ by $G_{\alpha\beta}$.

4.2 Equations of motion

There are conserved currents, called Noether currents, associated to each of the generators of the super Lie algebra $\mathfrak{psu}(2,2|4)$. The derivation of these Noether currents is given in Appendix B. The result obtained there is

$$J_N = J_1 + \star J_3. \tag{102}$$

The statement that these currents are conserved is

$$d \star J_N = d(\star J_1 + J_3) = 0. \tag{103}$$

These conservation equations hold as a consequence of the equations of motion. In fact, as in the case of the bosonic truncation, they are equivalent to the equations of motion that are obtained from arbitrary variations δX and $\delta\theta$. Note that the Noether current $J_1 + \star J_3$ is not flat, even though (as noted earlier) the Noether current for the bosonic truncation of the theory is flat.

In order to set the stage for the later proof of kappa symmetry, let us consider arbitrary variations of the Grassmann coordinates $\delta\theta$ (and $\delta\theta^\dagger$). Using Eqs. (49)–(53), this determines the variation of $\mathcal{A}$, which depends only on θ, to be

$$\delta\mathcal{A} = D\hat{\mathcal{M}} = d\hat{\mathcal{M}} + [\mathcal{A}, \hat{\mathcal{M}}], \tag{104}$$

where

$$\hat{\mathcal{M}} = \Gamma^{-1}\delta\Gamma = \mathcal{N} + R, \tag{105}$$

$$\mathcal{N} = (\hat{\mathcal{M}})_{\text{even}} = \begin{pmatrix} \mathcal{M} & 0 \\ 0 & \tilde{\mathcal{M}} \end{pmatrix}, \tag{106}$$

$$R = (\hat{\mathcal{M}})_{\text{odd}} = \begin{pmatrix} 0 & \zeta\rho \\ \zeta\rho^\dagger & 0 \end{pmatrix}. \tag{107}$$

Let us simultaneously vary the bosonic coordinates, which are encoded in X. We make a specific choice that is required to reveal local kappa symmetry, namely

$$\delta X = -(\mathcal{N}X + X\mathcal{N}^T). \tag{108}$$

Like X, the right-hand side is even and antisymmetric. Also, $\mathcal{N}^T = \mathcal{N}^{\overline{T}}$ is unambiguous.

Decomposing $\mathcal{A}$ into even and odd parts,

$$\mathcal{A} = \hat{K} - A_2, \tag{109}$$

Eq. (104) decomposes into the pair of equations

$$\delta\hat{K} = d\mathcal{N} + [\hat{K}, \mathcal{N}] + \hat{\Delta}, \tag{110}$$

and

$$\delta A_2 = -(DR)_{\rm odd} + [A_2, \mathcal{N}], \tag{111}$$

where

$$\hat{\Delta} = (DR)_{\rm even} = -[A_2, R]. \tag{112}$$

Let us now evaluate the variation of S_1, the first term in the Lagrangian, which is proportional to the integral of

$$\mathrm{str}(J_1 \wedge \star J_1) = \mathrm{str}(A_1 \wedge \star A_1). \tag{113}$$

To compute the variation of this expression, we require the variation of

$$A_1 = XdX^{-1} - \hat{K} - X\hat{K}^T X^{-1}. \tag{114}$$

Using the equations given above, one obtains

$$\delta A_1 = [A_1, \mathcal{N}] - \hat{\Delta} - X\hat{\Delta}^T X^{-1}. \tag{115}$$

Therefore,

$$\delta\, \mathrm{str}(A_1 \wedge \star A_1) = -2\, \mathrm{str}((\hat{\Delta} + X\hat{\Delta}^T X^{-1}) \wedge \star A_1) = -4\, \mathrm{str}(\hat{\Delta} \wedge \star A_1). \tag{116}$$

This gives

$$\delta\, \mathrm{str}(A_1 \wedge \star A_1) = 4\, \mathrm{str}([A_2, R] \wedge \star A_1) = 4\, \mathrm{str}(R[A_1 \wedge \star A_2 + \star A_2 \wedge A_1]). \tag{117}$$

In the last step we have used the identity $\star A_1 \wedge A_2 + A_1 \wedge \star A_2 = 0$. Altogether,

$$\delta S_1 = \frac{\sqrt{\lambda}}{4\pi} \int \mathrm{str}(R[A_1 \wedge \star A_2 + \star A_2 \wedge A_1]). \tag{118}$$

Turning to the WZ term, S_2, we need to compute the variation of

$$\mathrm{str}(J_2 \wedge J_3) = \mathrm{str}(A_2 \wedge A_3) = -i\, \mathrm{str}(A_2 \wedge X A_2^T X^{-1}). \tag{119}$$

Using Eq. (111),

$$\delta\, \mathrm{str}(J_2 \wedge J_3) = 2\, \mathrm{str}(\delta A_2 \wedge A_3) = -2\, \mathrm{str}(DR \wedge A_3), \tag{120}$$

where we put back the even part of DR, since it does not contribute to the supertrace. The variation δS_1 does not involve any derivatives of R, but the expression we have just found does contain one. Thus, if these two terms are to combine nicely, an integration by parts is required. The appropriate formula is

$$\mathrm{str}(DR \wedge A_3) = d\,\mathrm{str}(RA_3) - \mathrm{str}(RDA_3). \tag{121}$$

Applying the general rule $DA_i = \Gamma^{-1} dJ_i \Gamma$ to DA_3, and using the MC equation for dJ_3, we find

$$(DA_3)_{\mathrm{odd}} = -(A_3 \wedge A_1 + A_1 \wedge A_3), \tag{122}$$

and therefore

$$\delta\,\mathrm{str}(J_2 \wedge J_3) = -2\,\mathrm{str}(R[A_3 \wedge A_1 + A_1 \wedge A_3]) - 2d\,\mathrm{str}(RA_3). \tag{123}$$

Adjusting the normalization of the WZ term by setting $k = 2$, and dropping the total differential, gives the variation

$$\delta S_2 = -\frac{\sqrt{\lambda}}{4\pi} \int \mathrm{str}(R[A_3 \wedge A_1 + A_1 \wedge A_3]). \tag{124}$$

Combining Eqs. (118) and (124),

$$\delta S = \frac{\sqrt{\lambda}}{4\pi} \int \mathrm{str}(R[(\star A_2 - A_3) \wedge A_1 + A_1 \wedge (\star A_2 - A_3)]). \tag{125}$$

This implies that

$$(\star A_2 - A_3) \wedge A_1 + A_1 \wedge (\star A_2 - A_3) = 0 \tag{126}$$

is an equation of motion. Equivalently,

$$(\star J_2 - J_3) \wedge J_1 + J_1 \wedge (\star J_2 - J_3) = 0. \tag{127}$$

As we discussed earlier, the $h_{\alpha\beta}$ equation of motion implies that $h_{\alpha\beta}$ is proportional to $G_{\alpha\beta}$. Therefore, even though the Hodge dual was defined using $h_{\alpha\beta}$ in the preceding equations of motion, it can equivalently be defined using the induced metric $G_{\alpha\beta}$. Had we started with the $\sqrt{-G}$ form of S_1, we would have obtained the same equations of motion, but with the $G_{\alpha\beta}$ form of the Hodge dual in the first place.

In this section we have analyzed two types of variations. The first exploited the global symmetry of the theory to construct the corresponding conserved currents J_N, whose conservation encodes equations of motion. This entailed studying variations in which the infinitesimal $\mathfrak{psu}(2,2|4)$ parameters encoded in the supermatrix Λ are functions of the world-sheet

coordinates. The second variation we considered allowed an arbitrary local variation of the Grassmann coordinates $\delta\theta$ together with specific variations of the bosonic coordinates δX completely determined by $\delta\theta$. In the case of local Λ variations we found (in Appendix B) that the $d\Lambda$ part of the variations of the one-form supermatrices are concisely encoded in the single formula

$$\delta' A_+ = -\Gamma^{-1} d\Lambda\Gamma - X[\Gamma^{-1} d\Lambda\Gamma]^T X^{-1}. \tag{128}$$

Using Eqs. (111) and (115), one can deduce an analogous formula for the second type of variation

$$\delta' A_+ = -DR - X[DR]^T X^{-1}. \tag{129}$$

Using the identity $DR = \Gamma^{-1} d(\Gamma R\Gamma^{-1})\Gamma$, this suggests the correspondence

$$\Lambda \sim \Gamma R\Gamma^{-1}. \tag{130}$$

In other words, a special class of local Λ parameters are determined by $\delta\theta$ of the second type of variation. In terms of components, the correspondence is

$$\varepsilon(\sigma) = f^{-2}(\delta\theta + i\theta\delta\theta^\dagger\theta)\tilde{f}^{-2} \tag{131}$$

$$\omega(\sigma) = if^{-2}(\delta\theta\theta^\dagger - \theta\delta\theta^\dagger)f^{-2} \tag{132}$$

$$\tilde{\omega}(\sigma) = i\tilde{f}^{-2}(\delta\theta^\dagger\theta - \theta^\dagger\delta\theta)\tilde{f}^{-2}. \tag{133}$$

For these choices $R = \Gamma^{-1}\Lambda\Gamma$. Therefore the equations of motion (126) must actually be a special case of the conservation of the Noether current. In fact, Eq. (126) is equivalent to the odd part of the Noether current conservation equation written in the form $D(\star A_1 + A_3) = 0$.

4.3 Integrability

In general, for two matrices of one-forms J_1 and J_2 in 2d, with a Lorentzian signature metric,

$$\star J_1 \wedge J_2 + J_1 \wedge \star J_2 = 0. \tag{134}$$

Using this fact, Eq. (127) can be rewritten in the form

$$\star J_1 \wedge J_2 + J_2 \wedge \star J_1 + J_1 \wedge J_3 + J_3 \wedge J_1 = 0. \tag{135}$$

By taking the transpose of this equation and conjugating by X one deduces that

$$\star J_1 \wedge J_3 + J_3 \wedge \star J_1 + J_1 \wedge J_2 + J_2 \wedge J_1 = 0. \tag{136}$$

As has been explained, the last two equations are consequences of the conservation of the Noether current, $d(\star J_1 + J_3) = 0$. This equation and the MC equations (90)–(92) are the ingredients required for the proof of integrability given in [27]. Specifically, in terms of a spectral parameter x, the supermatrix of currents (or connections)

$$J(x) = c_1 J_1 + c_1' \star J_1 + c_2 J_2 + c_3 J_3 \tag{137}$$

is flat (*i.e.*, $dJ + J \wedge J = 0$) for

$$c_1 = -\sinh^2 x, \quad c_1' = \pm \sinh x \cosh x, \quad c_2 = 1 \mp \cosh x, \quad c_3 = \sinh x. \tag{138}$$

These equations allow one to construct an infinite family of conserved charges and establish integrability. The integrability of this theory was explored further in [28].

4.4 Kappa symmetry

The variation of the action in Eq. (125) is proportional to $\int W$, where

$$W = \mathrm{str}(R[C \wedge A_1 + A_1 \wedge C]) = \mathrm{str}([R, A_1] \wedge C), \tag{139}$$

and

$$C = \star A_2 - A_3. \tag{140}$$

In this section we will derive the local variations $\delta\theta$ and δX for which W vanishes, up to an exact two-form, thereby deriving the local kappa symmetry transformations.

As discussed earlier, the Hodge dual that appears in C can be defined using either the auxiliary metric $h_{\alpha\beta}$ or the induced metric $G_{\alpha\beta}$ depending on which form of the action is used. This choice does not matter for deriving classical equations of motion, since the h equation of motion relates $h_{\alpha\beta}$ to $G_{\alpha\beta}$. However, kappa symmetry is supposed to be a symmetry of the action, so equations of motion should not be invoked. The formulas in Appendix C, which will enable us to prove kappa symmetry, require using the induced metric $G_{\alpha\beta}$ to define the Hodge dual. Therefore, this will be the meaning of the Hodge dual for the remainder of this section and in Appendix C.

The supermatrix C has a special property, namely

$$C' = iXC^T X^{-1} = \star C, \tag{141}$$

since $XA_2^T X^{-1} = iA_3$ and $XA_3^T X^{-1} = iA_2$. This relationship, which is crucial for the proof of kappa symmetry, works because we have chosen the coefficient of the WZ term appropriately.

It is convenient is to decompose R and C into MW supermatrices. This means writing $R = R_1 + iR_2$ and $C = C_1 + iC_2$ such that

$$R' = iXR^T X^{-1} = R_1 - iR_2. \tag{142}$$

Since R is antihermitian, $R_1' = R_1$ is antihermitian and $R_2' = -R_2$ is hermitian. The decomposition of C works in the same way. Then Eq. (141) implies that

$$C_1 = \star C_1 \quad \text{and} \quad C_2 = - \star C_2. \tag{143}$$

Substituting these supermatrices, W takes the form

$$W = \mathrm{str}([R_1, A_1] \wedge C_1) - \mathrm{str}([R_2, A_1] \wedge C_2). \tag{144}$$

Next we invoke the identity derived in Appendix C

$$[R_i, A_1] = [\gamma(R_i), \star A_1], \tag{145}$$

and the duality properties of C_i given above, to deduce that

$$[R_1, A_1] \wedge C_1 = [\gamma_-(R_1), A_1] \wedge C_1 \tag{146}$$

and

$$[R_2, A_1] \wedge C_2 = [\gamma_+(R_2), A_1] \wedge C_2, \tag{147}$$

where we have introduced projection operators $\gamma_\pm = \frac{1}{2}(I \pm \gamma)$, so that

$$\gamma_\pm(R) = \frac{1}{2}\left(R \pm \gamma(R)\right). \tag{148}$$

Since $\gamma_+ \circ \gamma_- = \gamma_- \circ \gamma_+ = 0$, W vanishes and the action is invariant for the choices

$$\rho_1 = \gamma_+(\kappa) \quad \text{and} \quad \rho_2 = \gamma_-(\kappa), \tag{149}$$

where κ is an arbitrary (local) MW matrix. Since θ describes 32 real fermionic coordinates, this means that half of them are gauge degrees of freedom, which can be eliminated by a gauge choice. Recalling that $\rho = \rho_1 + i\rho_2 = f^{-1}\delta\theta\tilde{f}^{-1}$, we see that under a kappa symmetry transformation

$$\delta_\kappa\theta = f\big(\gamma_+(\kappa) + i\gamma_-(\kappa)\big)\tilde{f}. \tag{150}$$

The bosonic coordinates Y and Z are varied at the same time in the way described in Eqs. (108), (106). (See also Eqs. (51), (52), and (64).)

The superspace (x, θ) has $10 + 32$ dimensions. However, the local reparametrization and kappa symmetries imply that only $8 + 16$ of them induce independent off-shell degrees of

freedom of the superstring, just as in flat 10d spacetime. In the case of the flat spacetime theory, there is a gauge choice for which the superstring world-sheet theory becomes a free theory [26]. This is certainly not the case for the $AdS_5 \times S^5$ background, though some choices are more convenient than others. Possible gauge choices for the $AdS_5 \times S^5$ theory have been discussed extensively beginning with [8] [29] [30] [31]. This important issue will not be pursued here.

5 Conclusion

The superspace geometry of the $AdS_5 \times S^5$ solution of type IIB superstring theory and the dynamics of a fundamental superstring embedded in this geometry have been reexamined from a somewhat new perspective. We began by presenting a nonlinear realization of the superspace isometry supergroup $PSU(2,2|4)$ in terms of Grassmann coordinates only. The resulting formulas were interpreted as arising from a $PSU(2,2|4)/SU(2,2) \times SU(4)$ coset construction. Following that, unitary antisymmetric matrices $Z = \Sigma \cdot \hat{z}$ and $Y = \tilde{\Sigma} \cdot \hat{y}$ were introduced to describe the S^5 and AdS_5 coordinates, respectively. These matrices were interpreted as describing specific embeddings of S^5 inside $SU(4)$ and AdS_5 inside $SU(2,2)$.

Next we constructed three supermatrix one-forms J_1, J_2, and J_3 that transform linearly under infinitesimal global $\mathfrak{psu}(2,2|4)$ transformations, $\delta J_i = [\Lambda, J_i]$. In terms of these one-forms the superstring world-sheet action was shown to be

$$S = \frac{\sqrt{\lambda}}{16\pi} \int \left(\mathrm{str}(J_1 \wedge \star J_1) - 2\, \mathrm{str}(J_2 \wedge J_3) \right). \tag{151}$$

This action has manifest global $\mathfrak{psu}(2,2|4)$ symmetry and manifest local reparametrization invariance. The Hodge dual in the first term can be defined using either an auxiliary metric $h_{\alpha\beta}$ or the induced metric $G_{\alpha\beta} = -\frac{1}{4}(\mathrm{str} J_{1\alpha} J_{1\beta})$. However, the latter choice is required to establish local kappa symmetry, which is not manifest. Kappa symmetry was shown to arise from an interplay of three involutions. It determines the coefficient of the second term in the action, up to a sign that is convention dependent. Conservation of the $\mathfrak{psu}(2,2|4)$ Noether current $J_N = J_1 + \star J_3$ encodes the equations of motion. Using these equations, a one-parameter family of flat connections, required for the proof of integrability, was obtained.

All of these results are in complete agreement with what others have found long ago. So far, the main achievement of this work is to reproduce well-known results. However, the formulation described here has some attractive features that are not shared by previous ones. For one thing, the complete dependence of all quantities on the Grassmann coordinates is described by simple analytic expressions. Also, the action and the equations of motion have

manifest global $PSU(2,2|4)$ symmetry. In particular, at no point did we need to decompose $\mathfrak{psu}(2,2|4)$ into pieces,[10] as is often done.

There are two main directions that we hope to explore in the future using the results obtained here. One is to derive new facts about the dynamics of this fundamental superstring. The other is to explore other brane theories, such as a probe D3-brane embedded in the same $AdS_5 \times S^5$ background or a fundamental type IIA superstring in an $AdS_4 \times CP^3$ background.

Acknowledgments

The author acknowledges discussions with Matthew Heydeman as well as helpful comments from Warren Siegel and Edward Witten. This work was supported in part by the Walter Burke Institute for Theoretical Physics at Caltech and by U.S. DOE grant DE-SC0011632. The revision (v3) was completed in the summer of 2016 at the Aspen Center for Physics, which is supported by the National Science Foundation grant PHY-1066293.

[10] The generators associated to these pieces are usually denoted P, Q, D, J, R, S, K.

A Matrices for $SU(4)$ and $SU(2,2)$

In order to give an economical superspace description of $AdS_5 \times S^5$ and its $PSU(2,2|4)$ isometry, it is desirable to describe the bosonic coordinates and the bosonic subalgebra and in an appropriate way. It is well-known that the description of $SU(2)$ is very conveniently carried out using the three 2×2 Pauli matrices σ^a. This appendix will construct 4×4 matrices, Σ^a and Σ^m, that are convenient for describing $SU(4)$ and $SU(2,2)$.

In the case of $SU(4)$, we wish to define six antisymmetric 4×4 matrices $(\Sigma^a)^{\alpha\beta}$ and their hermitian conjugates $(\Sigma^{a\dagger})^{\bar{\alpha}\bar{\beta}}$. These matrices are invariant tensors of $SU(4)$ specifying how the six-vector representation couples to the antisymmetric Kronecker product of two four-dimensional representations $\mathbf{4} \times \mathbf{4}$ and $\bar{\mathbf{4}} \times \bar{\mathbf{4}}$, respectively. An essential difference from the case of $SU(2)$ is that the $\mathbf{6}$ is not the adjoint representation of $SU(4)$. (The latter arises in the Kronecker product $\mathbf{4} \times \bar{\mathbf{4}}$.) Another difference is that the $\mathbf{4}$ representation $SU(4)$ is complex, whereas the $\mathbf{2}$ representation of $SU(2)$ is pseudoreal. The invariant matrix $\eta_{\beta\bar{\beta}}$ is used to contract spinor indices in matrix products such as $\Sigma^a \eta \Sigma^{b\dagger}$. However, η is just the unit matrix I_4 in the case of $SU(4)$, so we can omit it without causing confusion. In the case of $SU(2,2)$, the matrix η is not the unit matrix, so we will display it in this appendix, but not in the main text.

We use the matrices Σ^a and $\Sigma^{a\dagger}$ to define 4×4 matrices

$$Z = \vec{\Sigma} \cdot \hat{z} \quad \text{and} \quad Z^\dagger = \vec{\Sigma}^\dagger \cdot \hat{z}. \tag{152}$$

The six-vector $\hat{z}$ describes a unit five-sphere, so $\hat{z} \cdot \hat{z} = 1$. We can encode a specific choice of the six antisymmetric matrices $(\Sigma^a)^{\alpha\beta}$ by introducing three complex coordinates $u = z^1 + iz^2$, $v = z^3 + iz^4$, and $w = z^5 + iz^6$ and defining[11]

$$Z^{\alpha\beta} = \begin{pmatrix} 0 & u & v & w \\ -u & 0 & -\bar{w} & \bar{v} \\ -v & \bar{w} & 0 & -\bar{u} \\ -w & -\bar{v} & \bar{u} & 0 \end{pmatrix}. \tag{153}$$

It is easy to verify that this choice satisfies

$$ZZ^\dagger = Z^\dagger Z = I_4, \tag{154}$$

which implies that Z is a unitary matrix.

The formulas given above imply that the Σ matrices satisfy the equations

$$(\Sigma^a \Sigma^{b\dagger} + \Sigma^b \Sigma^{a\dagger})^\alpha{}_\beta = 2\delta^{ab}\delta^\alpha_\beta \tag{155}$$

[11]This matrix and the one called Y (below) have appeared previously in the $AdS_5 \times S^5$ literature.

and

$$(\Sigma^{a\dagger}\Sigma^{b} + \Sigma^{b\dagger}\Sigma^{a})^{\bar{\alpha}}{}_{\bar{\beta}} = 2\delta^{ab}\delta^{\bar{\alpha}}_{\bar{\beta}}. \tag{156}$$

These imply, in particular, that

$$\mathrm{tr}(\Sigma^{a}\Sigma^{b\dagger} + \Sigma^{b}\Sigma^{a\dagger}) = 8\delta^{ab}. \tag{157}$$

One can also verify that

$$\frac{1}{2}\varepsilon_{\alpha\beta\gamma\delta}(\Sigma^{a})^{\gamma\delta} = (\Sigma^{a\dagger})_{\alpha\beta}, \tag{158}$$

which implies that

$$\frac{1}{2}\varepsilon_{\alpha\beta\gamma\delta}Z^{\gamma\delta} = Z^{\dagger}_{\alpha\beta}, \tag{159}$$

as expected. It is also interesting to note that

$$\det Z = (|u|^2 + |v|^2 + |w|^2)^2 = 1. \tag{160}$$

Thus, Z belongs to $SU(4)$, which means that Z parametrizes S^5 as a subspace of $SU(4)$. This is analogous to the equation $\sigma_2\vec{\sigma}\cdot\hat{x}$, discussed in the introduction, which describes S^2 as a subspace of $SU(2)$. The explicit formula for Z, given in Eq.(153), is never utilized. The purpose of presenting it is to demonstrate the existence of matrices Σ^a such that $Z = \vec{\Sigma}\cdot\hat{z}$ is an $SU(4)$ matrix.

In the introduction we interpreted the S^2 subspace of $SU(2)$ as an equivalence class of $SU(2)$ matrices. Therefore it is natural to seek the corresponding interpretation of the S^5 subspace of $SU(4)$. Since Z is an antisymmetric matrix, the appropriate equivalence relation is that two elements of $SU(4)$, g_0 and g_0', are equivalent if and only if there exists an element $g \in SU(4)$ such that $g_0' = g^T g_0 g$. For this choice of equivalence relation, the space of antisymmetric $SU(4)$ matrices forms an equivalence class, and the action of an arbitrary group element g on an element g_0 in this class is $g_0 \to g_0' = g^T g_0 g$. The action of the center of $SU(4)$, which is $\mathbb{Z}_4$, has a $\mathbb{Z}_2$ image. If g is i times the unit matrix, which is an element of the center, the map sends $g_0 \to -g_0$. So the isometry group is really $SO(6)$, as it should be. There are actually two S^5's inside $SU(4)$, which are distinguished by a change of sign in Eq. (159). The map $Z \to iZ$ is a one-to-one map relating the two spheres.

In the case of $SU(2,2)$ and AdS_5 we should redefine two of the six Σ matrices given above by a factor of i in order to incorporate the indefinite signature of $Spin(4,2)$. Therefore we modify the $SU(4)$ formulas accordingly and define $Y = \vec{\tilde{\Sigma}}\cdot\hat{y}$ by

$$Y^{\mu\nu} = \begin{pmatrix} 0 & iu & v & w \\ -iu & 0 & -\bar{w} & \bar{v} \\ -v & \bar{w} & 0 & -i\bar{u} \\ -w & -\bar{v} & i\bar{u} & 0 \end{pmatrix}. \tag{161}$$

Now, in the notation of Eq. (1), we make the identifications $u = y^0 + iy^5$, $v = y^1 + iy^2$, and $w = y^3 + iy^4$. Since $-y^2 = |u|^2 - |v|^2 - |w|^2 = 1$ describes the Poincaré patch of AdS_5, we see that the determinant of Y is unity. Next we take account of the indefinite signature of $SU(2,2)$ by defining

$$\eta^{\mu\bar{\nu}} = \eta_{\mu\bar{\nu}} = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix} = I_{2,2}. \tag{162}$$

Then, using this metric to contract spinor indices, one finds that

$$Y\eta Y^\dagger \eta = I_4, \tag{163}$$

where we use $-y^2 = |u|^2 - |v|^2 - |w|^2 = 1$ once again. This implies that Y is an element of $SU(2,2)$. Thus, just as in the compact case, we find that AdS_5 is represented as a subspace of the $SU(2,2)$ group manifold. Eq. (163) implies the algebra

$$(\Sigma^m \eta \Sigma^{n\dagger}\eta + \Sigma^n \eta \Sigma^{m\dagger}\eta)^\mu{}_\nu = -2\eta^{mn}\delta^\mu_\nu, \tag{164}$$

where η^{mn} is the $SO(4,2)$ metric.

The main text takes factors of η into account by only using "unbarred" indices, *i.e.,* by defining

$$Y^\dagger_{\mu\nu} = \eta_{\mu\bar{\mu}} Y^{\dagger\bar{\mu}\bar{\nu}} \eta_{\bar{\nu}\nu}. \tag{165}$$

Then we can write $YY^{-1} = I$, even though Y is not unitary.

Other interesting quantities are the connection one-forms for $\mathfrak{su}(4)$ and $\mathfrak{su}(2,2)$. The former is given by

$$\Omega_0 = ZdZ^\dagger = -dZZ^\dagger, \tag{166}$$

This matrix is antihermitian and traceless, which implies that it belongs to the $\mathfrak{su}(4)$ Lie algebra. To eliminate any possible doubt about this, we have computed the matrix explicitly:

$$\Omega_0 = \begin{pmatrix} ud\bar{u} + vd\bar{v} + wd\bar{w} & wdv - vdw & udw - wdu & vdu - udv \\ \bar{v}d\bar{w} - \bar{w}d\bar{v} & ud\bar{u} + \bar{v}dv + \bar{w}dw & ud\bar{v} - \bar{v}du & ud\bar{w} - \bar{w}du \\ \bar{w}d\bar{u} - \bar{u}d\bar{w} & vd\bar{u} - \bar{u}dv & vd\bar{v} + \bar{w}dw + \bar{u}du & vd\bar{w} - \bar{w}dv \\ \bar{u}d\bar{v} - \bar{v}d\bar{u} & wd\bar{u} - \bar{u}dw & wd\bar{v} - \bar{v}dw & wd\bar{w} + \bar{v}dv + \bar{u}du \end{pmatrix}. \tag{167}$$

Tracelessness is a consequence of $|u|^2 + |v|^2 + |w|^2 = 1$. This is a flat connection, since the two-form $d\Omega_0 + \Omega_0 \wedge \Omega_0$ vanishes. Similarly, the connection one-form

$$\tilde{\Omega}_0 = Y\eta dY^\dagger \eta = -dY\eta Y^\dagger \eta \tag{168}$$

belongs to the $\mathfrak{su}(2,2)$ Lie algebra, as it should. Moreover, $d\tilde{\Omega}_0 + \tilde{\Omega}_0 \wedge \tilde{\Omega}_0 = 0$, so it is also a flat connection.

To represent the Lie algebra of $\mathfrak{su}(4)$ we introduce the fifteen traceless antihermitian 4×4 matrices

$$(\Sigma^{ab})^{\alpha}{}_{\beta} = \frac{1}{2}(\Sigma^a\Sigma^{b\dagger} - \Sigma^b\Sigma^{a\dagger})^{\alpha}{}_{\beta}. \tag{169}$$

Similarly, for $\mathfrak{su}(2,2)$ we have

$$(\tilde{\Sigma}^{mn})^{\mu}{}_{\nu} = \frac{1}{2}(\tilde{\Sigma}^m\eta\tilde{\Sigma}^{n\dagger}\eta - \tilde{\Sigma}^n\eta\tilde{\Sigma}^{m\dagger}\eta)^{\mu}{}_{\nu}. \tag{170}$$

In this notation, the representations of the $\mathfrak{su}(4)$ and $\mathfrak{su}(2,2)$ Lie algebras are

$$\frac{1}{2}[\Sigma^{ab}, \Sigma^{cd}] = \delta^{bc}\Sigma^{ad} + \delta^{ad}\Sigma^{bc} - \delta^{ac}\Sigma^{bd} - \delta^{bd}\Sigma^{ac} \tag{171}$$

and

$$\frac{1}{2}[\tilde{\Sigma}^{mn}, \tilde{\Sigma}^{pq}] = \eta^{np}\tilde{\Sigma}^{mq} + \eta^{mq}\tilde{\Sigma}^{np} - \eta^{mp}\tilde{\Sigma}^{nq} - \eta^{nq}\tilde{\Sigma}^{mp}. \tag{172}$$

B Derivation of the Noether current

The Noether procedure for constructing the conserved current associated with a global symmetry instructs us to consider a *local* infinitesimal transformation, which is not a symmetry. The variation of the action then contains the derivative of the infinitesimal parameter times the Noether current. It then follows that conservation of the current is a consequence of the equations of motion. We wish to apply this procedure to the action in Eq. (101) by considering its variation under an arbitrary *local* $\mathfrak{psu}(2,2|4)$ transformation $\Lambda(\sigma)$ specified by the infinitesimal supermatrix

$$\Lambda = \begin{pmatrix} \omega & \zeta\varepsilon \\ \zeta\varepsilon^\dagger & \tilde{\omega} \end{pmatrix}. \tag{173}$$

The equations $\delta_\Lambda J_i = [\Lambda, J_i]$, which are correct when Λ is constant, need to be generalized to include additional terms depending on $d\Lambda$. We will also use the supermatrices $A_i = \Gamma^{-1}J_i\Gamma$, which were introduced in Sect. 3.5.

It will prove useful to know that

$$\Gamma^{-1}d\Lambda\Gamma = \begin{pmatrix} \chi + i\theta\phi^\dagger - i\phi\theta^\dagger + i\theta\tilde{\chi}\theta^\dagger & \zeta(\phi + \chi\theta - \theta\tilde{\chi} + i\theta\phi^\dagger\theta) \\ \zeta(\phi^\dagger + \tilde{\chi}\theta^\dagger - \theta^\dagger\chi + i\theta^\dagger\phi\theta^\dagger) & \tilde{\chi} + i\theta^\dagger\phi - i\phi^\dagger\theta + i\theta^\dagger\chi\theta \end{pmatrix}, \tag{174}$$

where

$$\chi = f^{-1}d\omega f^{-1}, \quad \tilde{\chi} = \tilde{f}^{-1}d\tilde{\omega}\tilde{f}^{-1}, \quad \text{and} \quad \phi = f^{-1}d\varepsilon\tilde{f}^{-1}. \tag{175}$$

We need to evaluate the variations of $\mathrm{str}(J_1 \wedge \star J_1)$ and $\mathrm{str}(J_2 \wedge J_3)$, which are the terms that appear in the action. The key to evaluating them is to rewrite them as $\mathrm{str}(A_1 \wedge \star A_1)$

and $\mathrm{str}(A_2 \wedge A_3)$ and to evaluate

$$\delta\,\mathrm{str}(A_1 \wedge \star A_1) = 2\,\mathrm{str}(\delta' A_1 \wedge \star A_1) \tag{176}$$

and

$$\delta\,\mathrm{str}(A_2 \wedge A_3) = 2\,\mathrm{str}(\delta' A_2 \wedge A_3), \tag{177}$$

Because these expressions have global $\mathfrak{psu}(2,2|4)$ symmetry, we only need to keep the terms involving $d\Lambda$ in δA_i. We have denoted these pieces by $\delta' A_i$. Calculating these, we find

$$\delta' A_1 = -[\Gamma^{-1} d\Lambda\Gamma]_{\mathrm{even}} - X[\Gamma^{-1} d\Lambda\Gamma]^T_{\mathrm{even}} X^{-1}, \tag{178}$$

$$\delta' A_2 = -[\Gamma^{-1} d\Lambda\Gamma]_{\mathrm{odd}}, \tag{179}$$

$$\delta' A_3 = iX[\Gamma^{-1} d\Lambda\Gamma]^T_{\mathrm{odd}} X^{-1}. \tag{180}$$

Since A_1 only contains diagonal blocks, which are even, so does its variation. These blocks are related to those in $\Gamma^{-1} d\Lambda\Gamma$, which is written out in Eq. (174), in the indicated fashion. Similarly, the variations of A_2 and A_3 are given by the odd off-diagonal blocks of $\Gamma^{-1} d\Lambda\Gamma$. We can add back the missing blocks in each case, since they do not contribute to the supertraces. Therefore

$$\begin{aligned}\delta\,\mathrm{str}(A_1 \wedge \star A_1) &= -2\,\mathrm{str}(\Gamma^{-1} d\Lambda\Gamma \wedge \star A_1) - 2\,\mathrm{str}(X[\Gamma^{-1} d\Lambda\Gamma]^T X^{-1} \wedge \star A_1)\\ &= -4\,\mathrm{str}(\Gamma^{-1} d\Lambda\Gamma \wedge \star A_1) = -4\,\mathrm{str}(d\Lambda \wedge \star J_1)\end{aligned} \tag{181}$$

and

$$\delta\,\mathrm{str}(A_2 \wedge A_3) = -2\,\mathrm{str}(\Gamma^{-1} d\Lambda\Gamma \wedge A_3) = -2\,\mathrm{str}(d\Lambda \wedge J_3). \tag{182}$$

Varying the combination that appears in the superstring action,

$$\delta\ (\mathrm{str}(A_1 \wedge \star A_1) + 2\,\mathrm{str}(A_2 \wedge A_3)) = -4\,\mathrm{str}(d\Lambda \wedge (\star J_1 + J_3)). \tag{183}$$

Thus, choosing the normalization, the Noether current that satisfies the conservation equation $d \star J_N = 0$ is

$$J_N = J_1 + \star J_3. \tag{184}$$

The formulas for $\delta' A_i$ imply that

$$\delta' A_+ = -\Gamma^{-1} d\Lambda\Gamma - X[\Gamma^{-1} d\Lambda\Gamma]^T X^{-1} \tag{185}$$

and

$$\delta' A_- = -\Gamma^{-1} d\Lambda\Gamma - X[\Gamma^{-1} d\Lambda\Gamma]^{\overline{T}} X^{-1}. \tag{186}$$

From these it follows that

$$\delta' J_+ = -d\Lambda - B_+ d\Lambda^T B_+^{-1} \quad \text{and} \quad \delta' J_- = -d\Lambda - B_- d\Lambda^{\overline{T}} B_-^{-1}, \tag{187}$$

where B_+ and B_- are defined in Eq. (88).

C Kappa symmetry projection operators

In order to figure out how kappa symmetry should work in the current context, it is very helpful to review the flat-space limit first. The flat-space theory was worked out using two 32-component MW spinors θ_i of the same chirality [26]. We will summarize the results in the spinor notation of section 5.2 of [25], without explaining that notation here. For an appropriate normalization constant k, it was shown that the variations are

$$\delta S_1 = k \int d^2\sigma \sqrt{-G} G^{\alpha\beta} (\partial_\alpha \bar{\theta}_1 \Pi_\beta \rho_1 + \partial_\alpha \bar{\theta}_2 \Pi_\beta \rho_2) \tag{188}$$

where $\Pi_\alpha = \Gamma_\mu \Pi_\alpha^\mu$, $G_{\alpha\beta} = \eta_{\mu\nu} \Pi_\alpha^\mu \Pi_\beta^\nu$, and $\rho_i = \delta\theta_i$. Similarly,

$$\delta S_2 = k \int d^2\sigma \varepsilon^{\alpha\beta} (\partial_\alpha \bar{\theta}_1 \Pi_\beta \rho_1 - \partial_\alpha \bar{\theta}_2 \Pi_\beta \rho_2). \tag{189}$$

Despite notational differences, it should be plausible that these equations describe the flat-space limit of the results founds in Sect. 5.1.

In this setting, the appropriate involution γ turned out to be $\gamma(\rho) = \gamma\rho$, where

$$\gamma = \frac{1}{2} \frac{\varepsilon^{\alpha\beta}}{\sqrt{-G}} \Pi_\alpha^\mu \Pi_\beta^\nu \Gamma_{\mu\nu}, \tag{190}$$

The formula $\gamma^2 = I$ is equivalent to

$$\frac{1}{2} \varepsilon^{\alpha\beta} \varepsilon^{\alpha'\beta'} \Pi_\alpha^\mu \Pi_\beta^\nu \Pi_{\alpha'}^\rho \Pi_{\beta'}^\lambda \{\Gamma_{\mu\nu}, \Gamma_{\rho\lambda}\} = -4GI. \tag{191}$$

To prove this, note that

$$\frac{1}{2} \{\Gamma_{\mu\nu}, \Gamma_{\rho\lambda}\} = (\eta_{\mu\lambda}\eta_{\nu\rho} - \eta_{\mu\rho}\eta_{\nu\lambda}) I + \Gamma_{\mu\nu\rho\lambda}, \tag{192}$$

but the last term does not contribute, because α, β, α', and β' only take two values.

Another useful identity, which is proved in a similar manner, is

$$\sqrt{-G} G^{\alpha\beta} \Pi_\beta \gamma = \varepsilon^{\alpha\beta} \Pi_\beta. \tag{193}$$

Multiplying on the right by γ and using $\gamma^2 = I$, it is also true that

$$\sqrt{-G} G^{\alpha\beta} \Pi_\beta = \varepsilon^{\alpha\beta} \Pi_\beta \gamma. \tag{194}$$

Substituting the latter identity into δS_1, one obtains

$$\delta S_1 + \delta S_2 = 2k \int d^2\sigma \varepsilon^{\alpha\beta} \Big(\partial_\alpha \bar{\theta}_1 \Pi_\beta \gamma_+ \rho_1 - \partial_\alpha \bar{\theta}_2 \Pi_\beta \gamma_- \rho_2 \Big), \tag{195}$$

where $\gamma_\pm = \frac{1}{2}(1 \pm \gamma)$ are projection operators. Thus, $\rho_1 = \delta\theta_1 = \gamma_- \kappa$ and $\rho_2 = \delta\theta_2 = \gamma_+ \kappa$ are 16 local symmetries. This means that half of the θ coordinates are gauge degrees of freedom of the string world-sheet theory.

The $AdS_5 \times S^5$ case

Kappa symmetry works in a similar way for the $AdS_5 \times S^5$ background geometry. The main challenge is to transcribe the flat-space formulas into the matrix notation used in this manuscript. The key equation is the defining equation of the involution γ. We claim that the correct counterpart of the operator γ in Eq. (190) is

$$\gamma(\rho) = -\frac{1}{2}\frac{\varepsilon^{\alpha\beta}}{\sqrt{-G}}\left(\Omega_\alpha\Omega_\beta\rho - 2\Omega_\alpha\rho'\tilde{\Omega}_\beta + \rho\tilde{\Omega}_\alpha\tilde{\Omega}_\beta\right). \tag{196}$$

This formula is unique up to sign ambiguities that are related to discrete symmetries of the world-sheet theory. In particular, the sign of the second term could be reversed. Using the definition in Eq. (70), $\gamma(\rho)$ satisfies the formula

$$[\gamma(\rho)]' = \gamma(\rho'). \tag{197}$$

Therefore, if ρ is a MW matrix, *i.e.,* $\rho = \rho'$, then $\gamma(\rho)$ is also a MW matrix. In general we can write $\rho = \rho_1 + i\rho_2$ and $\rho' = \rho_1 - i\rho_2$, where ρ_1 and ρ_2 are MW matrices. Substituting these expressions into Eq. (196), it is easy to see that the general case follows from the MW case. Therefore it is sufficient to prove that $\gamma \circ \gamma$ is the identity operator, *i.e.,* that γ is an involution, for the special case $\rho' = \rho$.

The fact that ρ is multiplied from both the left and the right is a bit awkward. Therefore let us recast Eq. (196) in a form with all multiplications acting from the left

$$-2\sqrt{-G}\gamma(\rho) = F\rho, \tag{198}$$

where

$$F = \varepsilon^{\alpha\beta}(\Omega_\alpha\Omega_\beta \otimes I - 2\Omega_\alpha \otimes \tilde{\Omega}_\beta^T + I \otimes \tilde{\Omega}_\beta^T\tilde{\Omega}_\alpha^T). \tag{199}$$

The second factor in the tensor products acts on the second index of the matrix ρ. In this notation the condition that γ is an involution is

$$F^2 = -4G(I \otimes I), \tag{200}$$

which we will now verify.

In the present problem $G_{\alpha\beta}$ is a sum of two terms, an S^5 part and an AdS_5 part,

$$G_{\alpha\beta} = g_{\alpha\beta} + \tilde{g}_{\alpha\beta}. \tag{201}$$

The crucial equations for verifying that $F^2 = -4G$ are

$$\{\Omega_\alpha, \Omega_\beta\} = -2g_{\alpha\beta}\, I \quad \text{and} \quad \{\tilde{\Omega}_\alpha, \tilde{\Omega}_\beta\} = 2\tilde{g}_{\alpha\beta}\, I. \tag{202}$$

These identities are established by utilizing equations analogous to Eq. (192) for the matrices introduced in Appendix A. They are consistent with Eq. (97). The determinant of $G_{\alpha\beta}$ is the sum of three pieces: $\det g_{\alpha\beta}$, $\det \tilde{g}_{\alpha\beta}$, and terms that are bilinear in $g_{\alpha\beta}$ and $\tilde{g}_{\alpha\beta}$. It is now straightforward to verify that upon squaring F the $\Omega^4 \otimes I$ terms give the $\det g$ piece, the $\Omega^2 \otimes \tilde{\Omega}^2$ terms give the mixed pieces, the $I \otimes \tilde{\Omega}^4$ terms give the $\det \tilde{g}$ piece. Furthermore, the $\Omega^3 \otimes \tilde{\Omega}$ and $\Omega \otimes \tilde{\Omega}^3$ terms vanish. Having established that $\gamma \circ \gamma = I$, we can define orthogonal projection operators $\gamma_\pm$ by

$$\gamma_+(\rho) = \frac{1}{2}[\rho + \gamma(\rho)] \quad \text{and} \quad \gamma_-(\rho) = \frac{1}{2}[\rho - \gamma(\rho)]. \tag{203}$$

The $AdS_5 \times S^5$ counterpart of Eq. (193) is

$$\sqrt{-G}G^{\alpha\beta}(\Omega_\beta \gamma(\rho') - \gamma(\rho)\tilde{\Omega}_\beta) = \varepsilon^{\alpha\beta}(\Omega_\beta \rho' - \rho\tilde{\Omega}_\beta). \tag{204}$$

This can be proved using Eqs. (196) and (202). Defining a pair of one-forms,

$$p = \Omega\gamma(\rho') - \gamma(\rho)\tilde{\Omega} \quad \text{and} \quad q = \Omega\rho' - \rho\tilde{\Omega}, \tag{205}$$

Eq. (204) can be recast in the more elegant form

$$p = \star q \quad \text{or} \quad q = \star p, \tag{206}$$

where the Hodge dual is defined using the induced metric $G_{\alpha\beta}$. This crucial identity, which is used to establish kappa symmetry in Sect. 4.4, relates three involutions: $\star$, μ (which maps $\rho \to \rho'$), and γ.

Let us now recast these results in terms of supermatrices A_1,

$$R = \begin{pmatrix} 0 & \zeta\rho \\ \zeta\rho^\dagger & 0 \end{pmatrix}, \tag{207}$$

and

$$\gamma(R) = \begin{pmatrix} 0 & \zeta\gamma(\rho) \\ \zeta\gamma(\rho)^\dagger & 0 \end{pmatrix}. \tag{208}$$

Corresponding to $\rho = \rho_1 + i\rho_2$ and $\rho' = \rho_1 - i\rho_2$, we can write $R = R_1 + iR_2$ and $R' = R_1 - iR_2$, where R_1 and R_2 satisfy $R_i = iXR_i^T X^{-1}$. For MW supermatrices, such as R_1 and R_2, Eqs. (205) and (206) combine to give

$$[R_i, A_1] = [\gamma(R_i), \star A_1] \quad i = 1, 2. \tag{209}$$

Together with the fact that γ_+ and γ_- are orthogonal projection operators, Eq. (209) is the key formula that is utilized in the proof of kappa symmetry in Sect. 4.4.

References

[1] J. M. Maldacena, “The Large-N Limit of Superconformal Field Theories and Supergravity,” Adv. Theor. Math. Phys. **2**, 231 (1998) [hep-th/9711200].

[2] M. B. Green and J. H. Schwarz, “Supersymmetrical String Theories,” Phys. Lett. B **109**, 444 (1982).

[3] L. Brink, J. H. Schwarz and J. Scherk, “Supersymmetric Yang-Mills Theories,” Nucl. Phys. B **121**, 77 (1977).

[4] R. Kallosh and A. Rajaraman, “Vacua of M Theory and String Theory,” Phys. Rev. D **58**, 125003 (1998) [hep-th/9805041].

[5] N. Beisert, C. Ahn, L. F. Alday, Z. Bajnok, J. M. Drummond, L. Freyhult, N. Gromov and R. A. Janik *et al.*, “Review of AdS/CFT Integrability: An Overview,” Lett. Math. Phys. **99**, 3 (2012) [arXiv:1012.3982 [hep-th]].

[6] G. 't Hooft, “A Planar Diagram Theory for Strong Interactions,” Nucl. Phys. B **72**, 461 (1974).

[7] R. R. Metsaev and A. A. Tseytlin, “Type IIB Superstring Action in AdS(5) x S**5 Background,” Nucl. Phys. B **533**, 109 (1998) [hep-th/9805028].

[8] R. Kallosh, J. Rahmfeld and A. Rajaraman, “Near Horizon Superspace,” JHEP **9809**, 002 (1998) [hep-th/9805217].

[9] I. Pesando, “A Kappa Fixed Type IIB Superstring Action on AdS(5) x S(5),” JHEP **9811**, 002 (1998) [hep-th/9808020].

[10] G. Arutyunov and S. Frolov, “Foundations of the AdS(5) x S**5 Superstring. Part I,” J. Phys. A **42**, 254003 (2009) [arXiv:0901.4937 [hep-th]].

[11] J. H. Schwarz, “Highly Effective Actions,” JHEP **1401**, 088 (2014) [arXiv:1311.0305 [hep-th]].

[12] J. H. Schwarz, “BPS Soliton Solutions of a D3-brane Action,” JHEP **1407**, 136 (2014) [arXiv:1405.7444 [hep-th]].

[13] S. R. Coleman, J. Wess and B. Zumino, “Structure of Phenomenological Lagrangians. 1.,” Phys. Rev. **177**, 2239 (1969).

[14] C. G. Callan, Jr., S. R. Coleman, J. Wess and B. Zumino, "Structure of Phenomenological Lagrangians. 2.," Phys. Rev. **177**, 2247 (1969).

[15] R. R. Metsaev, "On Manifest SU(4) Invariant Superstring Action in AdS(5) x S**5," Class. Quant. Grav. **18**, 1245 (2001) [hep-th/0012026].

[16] G. Mandal, N. V. Suryanarayana and S. R. Wadia, "Aspects of Semiclassical Strings in AdS(5)," Phys. Lett. B **543**, 81 (2002) [hep-th/0206103].

[17] N. Berkovits, M. Bershadsky, T. Hauer, S. Zhukov and B. Zwiebach, "Superstring Theory on AdS(2) x S**2 as a Coset Supermanifold," Nucl. Phys. B **567**, 61 (2000) [hep-th/9907200].

[18] N. Berkovits, "A New Limit of the AdS(5) x S**5 Sigma Model," JHEP **0708**, 011 (2007) [hep-th/0703282].

[19] N. Berkovits and C. Vafa, "Towards a Worldsheet Derivation of the Maldacena Conjecture," JHEP **0803**, 031 (2008) [AIP Conf. Proc. **1031**, 21 (2008)] [arXiv:0711.1799 [hep-th]].

[20] D. V. Volkov and V. P. Akulov, "Is the Neutrino a Goldstone Particle?," Phys. Lett. B **46**, 109 (1973).

[21] B. Zumino, "Nonlinear Realization of Supersymmetry in Anti de Sitter Space," Nucl. Phys. B **127**, 189 (1977).

[22] R. Roiban and W. Siegel, "Superstrings on AdS(5) x S**5 Supertwistor Space," JHEP **0011**, 024 (2000) [hep-th/0010104].

[23] P. Dai, R. N. Huang and W. Siegel, "Covariant Propagator in AdS(5) x S**5 Superspace," JHEP **1003**, 001 (2010) [arXiv:0911.2211 [hep-th]].

[24] W. Siegel, "AdS/CFT in Superspace," arXiv:1005.2317 [hep-th].

[25] K. Becker, M. Becker and J. H. Schwarz, "String Theory and M-Theory: A Modern Introduction," Cambridge, UK: Cambridge Univ. Pr. (2007)

[26] M. B. Green and J. H. Schwarz, "Covariant Description of Superstrings," Phys. Lett. B **136**, 367 (1984).

[27] I. Bena, J. Polchinski and R. Roiban, "Hidden Symmetries of the AdS(5) x S**5 Superstring," Phys. Rev. D **69**, 046002 (2004) [hep-th/0305116].

[28] L. F. Alday, G. Arutyunov and A. A. Tseytlin, "On Integrability of Classical Superstrings in AdS(5) x S**5," JHEP **0507**, 002 (2005) [hep-th/0502240].

[29] R. Kallosh, "Superconformal Actions in Killing Gauge," hep-th/9807206.

[30] R. Kallosh and J. Rahmfeld, "The GS String Action on AdS(5) x S**5," Phys. Lett. B **443**, 143 (1998) [hep-th/9808038].

[31] R. Kallosh and A. A. Tseytlin, "Simplifying Superstring Action on AdS(5) x S**5," JHEP **9810**, 016 (1998) [hep-th/9808088].

[32] G. Arutyunov and S. Frolov, "Integrable Hamiltonian for Classical Strings on AdS(5) x S**5," JHEP **0502**, 059 (2005) [hep-th/0411089].